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MATHEMATICS

Involutions of Compact Spaces and a Generalization of Borsuk's Theorem on Antipodes

by

J. W. JAWOROWSKI

Presented by K. BORSUK on May 2, 1955

1. In 1932 K. Borsuk proved three theorems about the n-dimensional sphere S_n, the most important of which is known under the name "Antipodensatz" (see [1], p. 178). These theorems are closely related to one another. The purpose of this paper is to give a generalization of them. The proof of our theorems will be based on the theorem which is announced by the author in [2].

2. Let M be a compact metric space. As in [2] we shall consider chains and cycles modulo 2 in M in the sense of Vietoris.

Let us suppose that the space M is p-acyclic modulo 2 and φ is a continuous involution *) of M. Then we can construct in M a sequence of true chains of M

$$\Gamma^p=(\underset{\sim}{\gamma}^{-1}, \underset{\sim}{x}^0, \underset{\sim}{\gamma}^0, \ldots, \underset{\sim}{x}^p, \underset{\sim}{\gamma}^p)$$

called a (p, φ) — *system in* M, which is defined as follows:

1^0 $\underset{\sim}{\gamma}^{-1}$ is the number 1 considered as a (-1)-dimensional true cycle of M.

2^0 For some r $(-1 \leqslant r < p)$ let an r-dimensional true cycle $\underset{\sim}{\gamma}^r$ of M be already defined such that $\varphi(\underset{\sim}{\gamma}^r)=\underset{\sim}{\gamma}^r$. Since $r<p$, the true cycle $\underset{\sim}{\gamma}^r$ is homologous to zero in M. Let $\underset{\sim}{x}^{r+1}$ be a true chain in M such that

$$\underset{\sim}{x}^{r+1}=\underset{\sim}{\gamma}^r.$$

Then we put

$$\underset{\sim}{\gamma}^{r+1}=\underset{\sim}{x}^{r+1}+\varphi(\underset{\sim}{x}^{r+1}).$$

Thus $\underset{\sim}{\gamma}^{r+1}$ is an $(r+1)$-dimensional true cycle in M and $\varphi(\underset{\sim}{\gamma}^{r+1})=\underset{\sim}{\gamma}^{r+1}$.

Let S_n be the unit sphere in the $(n+1)$-dimensional Euclidean space E_{n+1}, i.e. the set of points $p \in E_{n+1}$ which satisfy the condition $|p|=1$.

*) i. e. a mapping of M into itself, which satisfies the condition $\varphi(\varphi(x))=x$ for every $x \in M$.

Let α be the antipodal map of S_n, i. e. $\alpha(p)=-p$. Then α is a continuous involution of S_n. If M is a closed p-acyclic subset of S_n and $\alpha(M)=M$, then the (p, α)-system in M is the true antipodal p-system in M in the sense of [2]. The theorem, which is announced in [2], for $p=n$ and $A=S_n$, can be formulated as follows:

(A) *Let* $\Gamma^n=(\underset{\sim}{\gamma}^{-1}, \underset{\sim}{\varkappa}^0, \underset{\sim}{\gamma}^0, \ldots, \underset{\sim}{\varkappa}^n, \underset{\sim}{\gamma}^n)$ *be an* (n, α)*-system in* S_n. *Then the true cycle* $\underset{\sim}{\gamma}^n$ *is not homologous to zero in* S_n.

3. The main result of this paper is the following:

THEOREM 1. *Let* M *be an* n*-acyclic modulo* 2 *compact metric space and let* φ *be a continuous involution of* M. *Then every continuous mapping* f *of* M *into* S_n *which satisfies the condition* $f(\varphi(x))\neq f(x)$, *for every* $x\in M$, *maps the* k*-dimensional homology groups* $\boldsymbol{B}^k(M)$ *modulo* 2 *of* M *onto the* k*-dimensional homology groups* $\boldsymbol{B}^k(S_n)$ *modulo* 2 *of* S_n, *for every* $k=-1,0,1,2,\ldots$.

Proof. Since the homology groups $\boldsymbol{B}^k(S_n)$ of S_n are zero for every $k\neq n$, and the n-dimensional homology group $\boldsymbol{B}^n(S_n)$ contains only two elements, it remains to prove, that f maps some n-dimensional true cycle of M onto a true cycle of S_n, which is not homologous to zero in S_n.

Let $\Delta^n=(\underset{\sim}{\delta}^{-1}, \underset{\sim}{\lambda}^0, \underset{\sim}{\delta}^0, \ldots, \underset{\sim}{\lambda}^n, \underset{\sim}{\delta}^n)$ be a (n, φ)-system in M. Let us put

$$g(x)=\frac{f(x)-f(\varphi(x))}{|f(x)-f(\varphi(x))|}\ *)$$

for every $x\in M$. Since $f(x)\neq f(\varphi(x))$, g is a continuous mapping of M into S_n and satisfies the condition

$$g(\varphi(x))=-g(x)$$

for every $x\in M$. Hence g maps the (n, φ)-system Δ^n onto the (n, α)-system $g(\Delta^n)=(g(\underset{\sim}{\delta}^{-1}), g(\underset{\sim}{\lambda}^0), g(\underset{\sim}{\delta}^0), \ldots, g(\underset{\sim}{\lambda}^n), g(\underset{\sim}{\delta}^n))$ on S_n. From theorem (A) we conclude that the cycle $g(\underset{\sim}{\delta}^n)$ is not homologous to zero in S_n.

Furthermore, we observe that, for every $x\in M$, the points $f(x)$ and $g(x)$ are not antipodal. Indeed, if we suppose that $f(x)=-g(x)$, i. e.

$$f(x)=-\frac{f(x)-f(\varphi(x))}{|f(x)-f(\varphi(x))|},$$

then we obtain:

$$f(\varphi(x))=f(x)\cdot\left(1+|f(x)-f(\varphi(x))|\right). \tag{1}$$

Since $|f(x)|=|f(\varphi(x))|=1$, we conclude from (1) that $|1+|f(x)-f(\varphi(x))||=1+|f(x)-f(\varphi(x))|=1$, which is impossible, since $|f(x)-f(\varphi(x))|>0$.

The fact that, for every $x\in M$, the points $f(x)$ and $g(x)$ are not anti-

*) in the sense of operations with points in the space E_{n+1}.

podal, proves that the mappings f and g are homotopic (see, e.g. [7]). Hence the true cycles $f(\underline{\delta}^n)$ and $g(\underline{\delta}^n)$ are homologous in S_n. Since the true cycle $g(\underline{\delta}^n)$ is not homologous to zero in S_n, we conclude that $f(\underline{\delta}^n)$ is not homologous to zero in S_n either.

Therefore, the mapping f maps the n-dimensional true cycle $\underline{\delta}^n$ of M onto the n-dimensional true cycle $f(\underline{\delta}^n)$ of S_n, which is not homologous to zero in S_n. Thus Theorem 1 is proved.

Since the group $\boldsymbol{B}^n(S_n)$ is not trivial and since every continuous mapping f of M into S_n homopotic to a constant maps every true cycle $\underline{\gamma}^k$ lying in $\boldsymbol{M}$ onto a true cycle $f(\underline{\gamma}^k)$ homologous to zero in S_n, we deduce from Theorem 1 the following

COROLLARY 1. *Let M, φ and f be as in Theorem 1. Then f is not homotopic to a constant.*

As a simple consequence of Corollary 1, we obtain the following:

THEOREM 2. *Let M and φ be as in Theorem 1. Then, for every continuous mapping f of M into the Euclidean space E_n, there exists a point $x_0 \in M$, such that $f(\varphi(x_0))=f(x_0)$.*

For, the mapping f of M into E_n may be considered as a mapping of M into a proper subset of S_n, and hence f is homotopic to a constant. Then, if we suppose that $f(\varphi(x))\neq f(x)$ for every $x \in M$, we obtain a contradiction with Theorem 1, which proves Theorem 2.

From Theorem 2 and a lemma of Borsuk (see [1], p. 188) results:

THEOREM 3. *Let M and φ be as in Theorem 2 and let $M=M_0+M_1+\ldots+M_n$ be a decomposition of M into a sum of $n+1$ closed subsets of M. Then at least one of the sets M_i contains an involution pair $\{x, \varphi(x)\}$.*

If $M=S_n$ and φ is the antipodal map, Theorems 1, 2, and 3 reduce to Teorems I, II, and III of [1], respectively. Theorem III of [1], called the Theorem of Borsuk-Lusternik-Schnirelmann, was found and proved first by L. Lusternik and L. Schnirelmann [8] and later by Borsuk [1]. Concerning other analogous theorems see [3], [4], and [5].

If $M=S_n$ and φ is a continuous involution of S_n, we obtain Theorems I, II, and III of [1] generalized on the case in which the antipodal map of S_n is replaced by an arbitrary continuous involution. In particular, Theorem 1 yields

COROLLARY 2. *Let φ be a continuous involution of S_n and let f be a continuous mapping of S_n into itself such that $f(\varphi(x))\neq f(x)$ for every $x \in S_n$. Then the degree of f is odd.*

This theorem was proved recently by M. A. Krasnosyelsky in another way (see [6]).

4. THEOREM 4. *Let M be an acyclic modulo 2 *), compact metric space. Then any continuous involution φ of M has a fixed point.*

*) The space M is said to be acyclic modulo 2 if it is p-acyclic modulo 2 for every p.

Proof. We can admit (by the known theorem of Uıysohn [9]) that M is a subset of the Hilbert cube Q_ω. Let us suppose that $\varphi(x) \neq x$ for every $x \in M$. Since M is compact, there exists a number $\varepsilon > 0$ such that $\varrho(x, \varphi(x)) \geqslant 2\varepsilon$ for eveıy $x \in M$. Let r_n be the projection of Q_ω onto the Euclidean n-dimensional cube Q_n, defined for every $z = (z_1, z_2, \ldots) \in Q_\omega$ by the formula

$$r_n(z_1, z_2, \ldots) = (z_1, z_2, \ldots z_n, 0, 0, \ldots).$$

Then, for every $\eta > 0$ there exists a number n such that $\varrho(z, r_n(z)) < \eta$ for every $z \in Q_\omega$. Let n be such that

$$\varrho(z, r_n(z)) < \varepsilon.$$

Thus $f = r_n \varphi$ is a continuous mapping of M into the Euclidean cube Q_n and

$$\varrho(f(x), f(\varphi(x))) = \varrho(r_n(\varphi(x)), r_n(x)) \geqslant \varrho(\varphi(x), x) - \varrho(\varphi(x), r_n(\varphi(x))) - \varrho(x, r_n(x)) > 2\varepsilon - \varepsilon - \varepsilon = 0.$$

Hence $f(\varphi(x)) \neq x$ for every $x \in M$. But M is acyclic and therefore also n-acyclic and this contradicts Theorem 2. Thus Theorem 4 is proved.

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MATHEMATICS

Families of Acyclic Compacta in Euclidean n-Space

by

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Presented by K. BORSUK on April 25, 1955

1. We shall prove in this note some generalizations of the known theorems on sweeping and some new theorems on the same subject [1]; all this will be done in an unified treatment in which the Vietoris mapping theorem is the principal instrument.

2. **Terminology and notations.** We shall use the Vietoris homology theory. The letter m will always denote 0 or a natural number $\geqslant 2$. The group of integers reduced mod m will be denoted by $\mathfrak{G}_m$. The homomorphism induced on the r-th homology group $H^r(X, \mathfrak{G}_m)$ of the space X by the mapping $f\colon X \to Y$ will be denoted by $h^r_{f,m}$. A homomorphism h will be called *trivial* if the image of the whole group under h is a single element. A set X will be called *acyclic* mod m *in the dimension* r if $H^r(X, \mathfrak{G}_m)$ is trivial. If X is acyclic mod m in all dimensions it will be said simply to be *acyclic* mod m. The following theorem is of importance to us:

THE VIETORIS MAPPING THEOREM ([2] or [3]). *Let $f\colon X \to Y$ be a continuous mapping of a compact space X onto a space Y. If $f^{-1}(y)$ is acyclic* mod m *for every $y \in Y$, then $h^r_{f,m}$ is an isomorphism and is onto.*

Let X be a compact metric space and let $\Phi\colon X \to 2^Y$ be an upper semi-continuous *) mapping of X in the space 2^Y of all non-empty compact subsets of a (metric separable) space Y. The triple $\mathcal{F} = \{X, Y, \Phi\}$ will be called a *family*. The set X will be said to be the *basis* of $\mathcal{F}$, the sets $\Phi(x)$ — the *elements* of $\mathcal{F}$, the set $\bigcup_{x \in X} \Phi(x) \subset Y$ — the *field* of $\mathcal{F}$. The field will also be denoted by $\Phi(X)$. The family $\mathcal{F}$ will be said to be *simple* if its elements are disjoint; it will be said to be *acyclic* mod m if all its elements are acyclic mod m.

The subset $F = \underset{x,y}{E}[y \in \Phi(x)]$ of the Cartesian product $X \times Y$ will be said to be the *graph* of $\mathcal{F} = \{X, Y, \Phi\}$. The mapping $(x, y) \to x$ (resp. $(x, y) \to y$) of F in X (resp. in Y) will be called the *projection*.

*) For the definition of upper semi-continuous mapping see for instance [4], p. 32.

It is easily seen that the graph and the field of a family are compact sets.

Let $\mathcal{F}_i=\{X, Y, \Phi_i\}$, $i=1,2$, be two families. We shall call the family $\mathcal{F}_1$ a *prolongation* of the family $\mathcal{F}_2$ if $\Phi_1(x)\supset\Phi_2(x)$ for every $x\in X$. The class of all families acyclic mod m which are prolongations of a given family $\mathcal{F}$ will be denoted by $\mathfrak{P}_m(\mathcal{F})$.

Let $\mathcal{F}=\{X, E_n, \Phi\}$ be a family with the field in the Euclidean n-space E_n. The subclass of all prolongations of $\mathcal{F}$ consisting of all those families whose field contains all bounded components of $E_n-\Phi(X)$ will be denoted by $\mathfrak{S}(\mathcal{F})$. In the investigation of $\mathfrak{S}(\mathcal{F})$ the following lemma will be useful:

LEMMA. *Let $\mathcal{F}_i=\{X, E_n, \Phi_i\}$, $i=1,2$, be two families and suppose that $\mathcal{F}_1$ is a prolongation of $\mathcal{F}_2$. Let $j\colon \Phi_2(X)\to\Phi_1(X)$ be the identity mapping. If the homomorphism $h^{n-1}_{j,m}$ is trivial for some m then $\mathcal{F}_1\in\mathfrak{S}(\mathcal{F}_2)$.*

Proof. Suppose that $\mathcal{F}_1\in\mathfrak{S}(\mathcal{F}_2)$. Then there exists a bounded component G of $E_n-\Phi_2(X)$ such that $G-\Phi_1(X)\neq 0$. Let $p_1\in G-\Phi_1(X)$ and let p_2 belong to the unbounded component of $E_n-\Phi_1(X)$. For every m, the pair (p_1, p_2) may be considered as an algebraic 0-dimensional cycle z mod m. Since this cycle is not homologous to 0 in $E_n-\Phi_2(X)$ there exists in $\Phi_2(X)$ an $(n-1)$-dimensional true convergent cycle γ mod m linked mod m with the cycle z (see [5] p. 185). Thus, γ is not homologous to 0 in $\Phi_1(X)$ and we infer that for every m the homomorphism $h^{n-1}_{j,m}$ is not trivial.

3. Prolongations of simple families. If the family $\mathcal{F}=\{X, Y, \Phi\}$ is simple then its elements constitute an upper semi-continuous decomposition of $\Phi(X)$ and the space X may be considered as the space of this decomposition. The natural mapping of $\Phi(X)$ onto X will be denoted by Φ^{-1}.

THEOREM 1. *Let $\mathcal{F}=\{X, E_n, \Phi\}$ be a simple family. If the homomorphism $h^{n-1}_{\Phi^{-1},m}$ is trivial then $\mathfrak{P}_m(\mathcal{F})\subset\mathfrak{S}(\mathcal{F})$.*

Proof. Let $\mathcal{F}_1\in\mathfrak{P}_m(\mathcal{F})$ and let F, F_1 be the graphs of $\mathcal{F}, \mathcal{F}_1$ respectively. Since $\mathcal{F}$ is simple, the projection of F onto $\Phi(X)$ is a homeomorphism. Denote its inversion by p, the projection of F_1 onto $\Phi_1(X)$ by p_1, the projection of F_1 onto X by q, the identity mapping $F\to F_1$ by i, the identity mapping $\Phi(X)\to\Phi_1(X)$ by j and consider the following diagram

$$\begin{array}{ccccc} F & \xrightarrow{i} & F_1 & \xrightarrow{q} & X \\ \uparrow{\scriptstyle p} & & \downarrow{\scriptstyle p_1} & & \\ \Phi(X) & \xrightarrow[j]{} & \Phi_1(X). & & \end{array}$$

It is easily seen that $qip=\Phi^{-1}$, $p_1ip=j$ and that the inverse-image of any point of X under q is acyclic. Hence by the Vietoris mapping theorem $h^{n-1}_{q,m}$ is an isomorphism and if the homomorphism $h^{n-1}_{\Phi^{-1},m}=$

$h^{n-1}_{q,m}\, h^{n-1}_{i,m}\, h^{n-1}_{p,m}$ is trivial then $h^{n-1}_{i,m}\, h^{n-1}_{p,m}$ is also. Thus the homomorphism $h^{n-1}_{j,m} = h^{n-1}_{p,,m}\, h^{n-1}_{i,m}\, h^{n-1}_{p,m}$ is trivial and the theorem follows at once from the lemma.

Applications. a) Let $M \subset E_n$ be a compact $(n-1)$-dimensional manifold. Suppose M is a fibre space with the fibre Y of dimension >0, the basis X, the projection p (see [6], § 1.1). For the family $\mathcal{F}=\{X, E_n, p^{-1}\}$ we have $\bigcup_m \mathfrak{P}_m(\mathcal{F}) \subset \mathfrak{S}(\mathcal{F})$.

For in this case $\dim X < n-1$, hence X is acyclic mod m in dimension $n-1$ for every m.

b) Let $M \subset E_n$ be a compact set and f its continuous mapping onto a space X acyclic mod m in the dimension $n-1$. For the family $\mathcal{F}= =\{X, E_n, f^{-1}\}$ we have $\mathfrak{P}_m(\mathcal{F}) \subset \mathfrak{S}(\mathcal{F})$.

c) Let $M \subset E_n$ be an $(n-1)$-dimensional pseudomanifold and T the simplicial complex of its triangulation. Let $g\colon M \to M$ be a continuous involution on M (i. e. $gg(x)=x$) without fixed points and permuting the simplexes of T. Denote by M^* the space obtained from M by identification of points x and $g(x)$ and let $f\colon M \to M^*$ be the identification mapping. For the family $\mathcal{F}=\{M^*, E_n, f^{-1}\}$ we have $\mathfrak{P}_2(\mathcal{F}) \subset \mathfrak{S}(\mathcal{F})$.

To prove this, denote by γ the generator of $H^{n-1}(M, \mathfrak{G}_2)$. The generator γ is represented by the chain of all simplexes of M, each with coefficient 1. Hence $f(\gamma)$ consists of all simplexes of M^* each with coefficient 2. Thus $f(\gamma) \sim 0$ in M^* mod 2. Hence $h^{n-1}_{f,2}$ is trivial.

This last theorem will be generalized in a forthcoming note by A. Kosiński.

4. The general case. Let $\mathcal{F}=\{X, E_n, \Phi\}$ be a family and let F be its graph. Denote the projection of F onto X by f_1, into E_n — by f_2.

THEOREM 2. *If the homomorphism $h^{n-1}_{f_1,m}$ is trivial and $h^{n-1}_{f_2,m}$ is a homomorphism onto the corresponding group of $\Phi(X)$ then $\mathfrak{P}_m(\mathcal{F}) \subset \mathfrak{S}(\mathcal{F})$.*

The proof differs very little from that of Theorem 1 (which is a special case of Theorem 2) and will be omitted.

Applications. a) Consider a family of great $(n-k-1)$-dimensional spheres orthogonal to a given k-dimensional great sphere S $(k<n-1)$ on the $(n-1)$-sphere $S_{n-1} \subset E_n$. Any such sphere intersects S in two antipodal points and distinct spheres intersect S in distinct pairs of points. Denote by P_k the projective k-dimensional space obtained from S by identification of antipodal points. Let $i\colon S \to P_k$ be the identification mapping. Define the family $\mathcal{F}=\{P_k, E_n, \Phi\}$ as follows: $\Phi(x)=$ the $(n-k-1)$-dimensional great sphere orthogonal to S and intersecting S in $i^{-1}(x)$. Then $\mathfrak{P}_m(\mathcal{F}) \subset \mathfrak{S}(\mathcal{F})$.

Consider namely the graph F of $\mathcal{F}$. The set F is an $(n-1)$-dimensional manifold (it is easily seen that $F = S_{n-k-1} \times P_k$). Since $k<n-1$ the ho-

momorphism $h^{n-1}_{f_1,m}$ is trivial for any m. Since f_2 maps F onto S_{n-1} with degree 1 it follows that $h^{n-1}_{f_2,m}$ is a homomorphism onto.

b) Let $M_{n,k}$ be the Grassman manifold of all (unoriented) k-dimensional planes through the origin of E_n (see [6], § 7.9). Let S_{n-1} be the unit sphere of E_n. If $x \in M_{n,k}$ represents the plane π, then let $\Phi(x)$ be the intersection of π and S_{n-1}. Let $\mathcal{F}=\{M_{n,k}, E_n, \Phi\}$. It follows immediately from the preceding example that if $k>1$, then $\bigcup_m \mathfrak{P}_m(\mathcal{F}) \subset \mathfrak{S}(\mathcal{F})$. (The case $k=1$ was investigated in example c) in 3).

5. Common prolongation of two families.

THEOREM 3. *Let $\mathcal{F}=\{X, E_n, \Phi\}$ be a simple family, let $\mathcal{F}_1=\{X, E_n, \Phi_1\}$ be a family with field acyclic* mod m *in the dimension $n-1$ and let f be the projection of the graph F_1 of $\mathcal{F}_1$ onto X. Suppose that $h^{n-1}_{f,m}$ is a homomorphism onto. Then $\mathfrak{P}_m(\mathcal{F}) \cap \mathfrak{P}_m(\mathcal{F}_1) \subset \mathfrak{S}(\mathcal{F})$.*

Proof. Let $\mathcal{F}_2=\{X, E_n, \Phi_2\}$ be a common prolongation of $\mathcal{F}$ and $\mathcal{F}_1$ acyclic mod m. Let F, F_1, F_2 be the corresponding graphs, p — the projection of F_2 onto X, q — the projection of F_2 onto $\Phi_2(X)$. Since $\mathcal{F}$ is simple, the mapping q on F is a homomorphism. Let γ be a true convergent $(n-1)$-dimensional cycle mod m in $\Phi(X)$ and let z be a cycle in F_1 such that $p(z) \sim pq^{-1}(\gamma)$ in X. The Vietoris mapping theorem implies that $z \sim q^{-1}(\gamma)$ in F_2, hence $q(z) \sim \gamma$ in $\Phi_2(X)$. But $q(z)$ is a cycle in $\Phi_1(X)$ hence $q(z) \sim 0$ in $\Phi_1(X)$ and this implies $\gamma \sim 0$ in $\Phi_2(X)$. The theorem itself follows then from the lemma of 2.

Applications. a) Let $\mathcal{F}=\{X, E_n, \Phi\}$ be a simple family and let $\mathcal{F}_1=\{X, E_n, \Phi_1\}$ be a family acyclic mod m with field acyclic mod m in the dimension $n-1$. Then $\mathfrak{P}_m(\mathcal{F}) \cap \mathfrak{P}_m(\mathcal{F}_1) \subset \mathfrak{S}(\mathcal{F})$.

b) Let $\mathcal{F}=\{X, E_n, \Phi\}$ be a simple family and let $\mathcal{F}_1=\{X, E_n, \Phi_1\}$ be a family with the field acyclic mod m. Suppose, moreover, that there exists a mapping $f\colon X \to \Phi_1(X)$ satisfying $f(x) \in \Phi_1(x)$ for every $x \in X$. Then $\mathfrak{P}_m(\mathcal{F}) \cap \mathfrak{P}_m(\mathcal{F}_1) \subset \mathfrak{S}(\mathcal{F})$.

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MATHEMATICS

A Remark on the Mixing Theorem

by
C. RYLL-NARDZEWSKI

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Let (X, B, m) be a σ-measure space with $m(X)=1$. We suppose that a fixed transformation T of X into itself is measurable and measure-preserving (i. e. if $E \in B$, then the inverse image $T^{-1}E \in B$ and $m(T^{-1}E)=m(E)$). As usual, we say that T is *indecomposable*, if every almost T-invariant set E (i. e. such that $m[(E-T^{-1}E)+(T^{-1}E-E)]=0$) is of measure zero or one. Obviously, a transformation T is indecomposable if every almost T-invariant function f (i. e. such that $f(Tx)=f(x)$ almost everywhere) *) is constant a. e.

T is called *weakly mixing* (cf. Hopf [1]) if for any $E, F \in B$

$$\lim_{j \text{ non} \in N} m(E \cdot T^{-j}F)=m(E)m(F),$$

where N is a suitable sequence of density zero.

Every weakly mixing transformation T is indecomposable.

The following equivalence is known for $1-1$ transformations:

THEOREM. *Weak mixing is equivalent to each of the following conditions*:

(a) *The square of* T:

$$T \times T(x, y)=(Tx, Ty)$$

is indecomposable with respect to the direct product measure $m \times m$.

(b) *The unique proper function*:

$$f(Tx)=\lambda f(x) \qquad (\lambda - \textit{complex})$$

is the function $f(x)=\text{const}$.

The purpose of this paper is *to prove this theorem without the hypothesis that* T *is* $1-1$.

*) In the sequel, the equality sign between two functions will always denote equality a. e.

It suffices to show that (b) implies (a), because in the known proofs of the remaining implications (see Hopf [1]) this hypothesis is not essential.

Assume that the transformation $T \times T$ is decomposable and let $\chi(x, y)$ denote the characteristic function of an almost invariant set $E \subset X \times Y$ with $0 < m \times m(E) < 1$.

It is easy to see that one of the functions $\chi(x, y) \pm \chi(y, x)$ is not constant a. e. We set respectively

$$K(x, y) = \chi(x, y) + \chi(y, x)$$

or

$$K(x, y) = i(\chi(x, y) - \chi(y, x)).$$

We have

$$K(y, x) = \overline{K}(x, y) \quad \int\int |K(x, y)|^2 \, dx\, dy < \infty \tag{1}$$

$$K(Tx, Ty) = K(x, y). \tag{2}$$

Further we treat K as the kernel of the integral equation

$$\mu \int K(x, y) f(y)\, dy = f(x)$$

in the space $L^2(m)$ and, in view of (1), we can apply Hilbert-Schmidt theorems (see e. g. Riesz-Nagy [2], p. 239–242).

Let $L^2\mu$ be the subspace of $L^2(m)$ consisting of all proper functions corresponding to the proper value μ.

If a function $f(x)$ belongs to $L^2\mu$ then the function $f(Tx)$ also does. In fact

$$f(Tx) = \mu \int K(Tx, y) f(y)\, dy = \mu \int K(Tx, Ty) f(Ty)\, dy = \mu \int K(x, y) f(Ty)\, dy.$$

Obviously, $\int |f(Tx)|^2 dx = \int |f(x)^2| dx$. Thus, the correspondence between functions: $f(x) \to f(Tx)$ is a linear isometric transformation of $L^2\mu$ onto itself ($L^2\mu$ is of finite dimension). Then, there is an orthonormal basis of $L^2\mu$, consisting of the proper functions $f_1, f_2, \ldots, f_n$ of the transformation T:

$$f_j(Tx) = \lambda_j f_j(x) \qquad |\lambda_j| = 1 \qquad j = 1, 2, \ldots, n.$$

Since K is not constant a. e., there is a proper value μ and a function f_j also not constant a. e.

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MATHEMATICS

A Proof of ε-Theorems

by
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Presented by K. KURATOWSKI on April 30, 1955

The aim of this paper *) is to outline the idea of a simple algebraic proof of the well-known ε-theorems **). The proof is to appear in Fundamenta Mathematicae (vol. 43) in a paper under the same title.

The method applied is the same as the method used in the previous papers, e. g. in *Algebraic models of axiomatic theories* cited hereafter as [A M] (see [4]). The knowledge of [A M] is here assumed.

The proof sketched in this note is valid also for non-enumerable theories. It is known that the algebraic proof of the Skolem-Gödel theorem for non-enumerable open theories can be carried out in a very simple way (see [3] or [5] and [6]). The transition to the case of arbitrary theories may be accomplished by the use of the ε-theorem and was applied by Łoś [3]. Thus, Malcev's theorem i. e. the Skolem-Gödel theorem for non-enumerable theories may be proved in a simple manner, different from that of Henkin [1].

It should also be noted that the suggested proof of ε-theorems can be generalized for theories formalized on the basis of certain non-classical functional calculi.

Let $T(\mathfrak{A})$ ***) be an elementary theory (with the set $\mathfrak{A}$ of axioms) formalized on the basis of the functional calculus $T(\Lambda)$ ****). We assume that the following symbols are the primitive signs of $T(\mathfrak{A})$: *individual variables* $x_1, x_2, \ldots$; *individual constants* $a_1, a_2, \ldots$; *functors with l-arguments* $f_1^l, f_2^l, \ldots$ (i. e. symbols for functions from individuals to individuals); *predicates with k-arguments* $F_1^k, F_2^k, \ldots$; *sentential operators* $+, ., \rightarrow, -$; *quantifiers* $\prod_{x_i}, \sum_{x_i}$ $(i = 1, 2, \ldots)$. The set $Cn(\mathfrak{A})$ of all *theorems* of $T(\mathfrak{A})$ consists

*) Presented at the Polish Mathematical Society on March 11, 1955.

**) See Hilbert-Bernays [2]. Another simple proof of the first ε-theorem was given by Łoś [3].

***) For the description of the theory $T(\mathfrak{A})$ see [A M], p. 298.

****) Λ denotes always the empty set.

of all $\alpha \in \mathfrak{A}$, of all substitutions of tautologies of the sentential calculus and of all other formulas derivable from the above mentioned one, by the use of the known rules of inference.

If the set $\mathfrak{A}$ contains only open formulas, we shall distinguish the set $Cn_0(\mathfrak{A})$ ($Cn_1(\mathfrak{A})$) of theorems formally proved without using the rules of inference for the quantifiers (and without using the rule of substitution for individual variables). The theory with the set $Cn_0(\mathfrak{A})$ of theorems will be denoted by $T_0(\mathfrak{A})$. In this case let $\bar{\bar{\mathfrak{A}}}$ denote the smallest class of formulas of $T(\mathfrak{A})$ containing $\mathfrak{A}$ and closed with respect to substitutions. It is known that:

1. $Cn_1(\bar{\bar{\mathfrak{A}}}) = Cn_0(\mathfrak{A})$.

2. If $\alpha \in Cn_1(\bar{\bar{\mathfrak{A}}})$ *then there exist* $\beta_1, \ldots, \beta_n \in \bar{\bar{\mathfrak{A}}}$ such that $\beta_1 \cdot \ldots \cdot \beta_n \to \alpha \in Cn(\wedge)$ (i. e. $\beta_1 \cdot \ldots \cdot \beta_n \to \alpha$ is provable in $T(\wedge)$).

Given a theory $T(\mathfrak{A})$ we shall treat each formula α of this theory as an algebraic functional $(J, B)\Phi_\alpha$ *) defined in a domain $J \neq \wedge$ of individuals, with values belonging to a fixed complete Boolean algebra B. Instead of $(J, B)\Phi_\alpha$ we shall also write $(J, B)\Phi(\alpha)$.

Let

$$(*) \qquad a_i = j_i \in J, \qquad f_n^l = \tau_n^l, \qquad F_m^k = \varphi_m^k$$

(where τ_n^l are l-argument functions defined on J with values in J and φ_m^k are k-argument functions defined on J with values in B) be a fixed system of valuations of all individual constants, functions and predicates of $T(\mathfrak{A})$. This system will be denoted by $\mathfrak{M} = [\{j_i\}, \{\tau_n^l\}, \{\varphi_m^k\}]$ and will be called an algebraic *pseudomodel* of $T(\mathfrak{A})$ in J and B. $\mathfrak{M}$ is said to be an *algebraic generalized model of* $T(\mathfrak{A})$ ($T_0(\mathfrak{A})$) if for every $\alpha \in \mathfrak{A}$, $(\mathfrak{M})\Phi_\alpha = 1$, i. e. if the value of $(J, B)\Phi_\alpha$ is equal to the unit of B for the values of its arguments fixed by (*) and for any values of $x_i (i = 1, 2, \ldots)$.

It is known (see [A M] 3.5) that

3. *If* $\alpha \in Cn(\mathfrak{A})$ ($Cn_0(\mathfrak{A})$) *then* $(\mathfrak{M})\Phi_\alpha = 1$ *in every generalized model* $\mathfrak{M}$ *in every domain* $J \neq \wedge$ *and every complete Boolean algebra* B. *In particular if* $\alpha \in Cn(\wedge)$ *then* $(\mathfrak{M})\Phi_\alpha = 1$ *in every pseudomodel* $\mathfrak{M}$.

The following theorem is a consequence of Theorem 3.6 proved in [A M].

4. *Given a theory* $T(\mathfrak{A})$ ($T_0(\mathfrak{A})$), *there exists a functionally free generalized algebraic model* $\mathfrak{N}$ ($\mathfrak{N}_0$) *of* $T(\mathfrak{A})$ ($T_0(\mathfrak{A})$), i. e. *with the following property: for any formula* α, $\alpha \in Cn(\mathfrak{A})$ ($\alpha \in Cn_0(\mathfrak{A})$) *if and only if*, $(\mathfrak{N})\Phi_\alpha = 1$ ($(\mathfrak{N}_0)\Phi_\alpha = 1$).

*) For the description of the functional $(J, B)\Phi_\alpha$ see [A M], p.297.

5. *The first ε-theorem* *). *Given an open formula a of a theory $T(\mathfrak{A})$ all of whose axioms are open formulas, if $a \in Cn(\mathfrak{A})$ then $a \in Cn_0(\mathfrak{A})$.*

Indeed, suppose that $a \in Cn_0(\mathfrak{A})$. Then by 4, $(\mathfrak{N}_0)\Phi_a = 1$. But $\mathfrak{N}_0$ is also an algebraic generalized model of $T(\mathfrak{A})$. Hence, on account of 3, $a \in Cn(\mathfrak{A})$.

In the sequel let $T(\mathfrak{A})$ be a theory, all of whose axioms $a_\varrho \in \mathfrak{A} (\varrho \in R)$ are closed formulas in the prenex normal form. Using the known *Skolem method* (see [2]) of elimination of quantifiers it is possible to pass from $T(\mathfrak{A})$ to a theory $T^*(\mathfrak{A}^*)$ with the set $\mathfrak{A}^*$ of open formulas, and with new primitive signs of annexed individual constants and functors. Then $T^*(\wedge)$ denotes the functional calculus, being the basis for the formalization of $T^*(\mathfrak{A}^*)$. It is easy to see that

6. *Given a formula a of $T(\mathfrak{A})$, if a is provable in $T^*(\wedge)$, then it is also provable in $T(\wedge)$.*

For any $a_\varrho \in \mathfrak{A}$, we shall denote by γ_ϱ the arbitrary substitution of the formula $a_\varrho^* \in \mathfrak{A}^*$ arising from a_ϱ by the elimination of quantifiers.

Given a pseudomodel $\mathfrak{M}$ of $T(\mathfrak{A})$ in J and B, we may extend it to a pseudomodel $\mathfrak{M}^*$ of $T^*(\mathfrak{A}^*)$ in J and B, fixing a valuation for all annexed signs.

7. *Given a pseudomodel $\mathfrak{M}$ of the theory $T(\mathfrak{A})$ and formulas $\gamma_{\varrho_1}, \ldots, \gamma_{\varrho_r}$ if for every extension $\mathfrak{M}^*$ of the pseudomodel $\mathfrak{M}$, $(\mathfrak{M}^*)\Psi(\gamma_{\varrho_1} \cdot \ldots \cdot \gamma_{\varrho_r}) = 0$ identically, then $(\mathfrak{M})\Phi(a_{\varrho_1} \cdot \ldots \cdot a_{\varrho_r}) = 0$.*

The second ε-theorem is equivalent (see [2]) with the following.

8. *The consistency of a theory $T(\mathfrak{A})$ implies the consistency of $T^*(\mathfrak{A}^*)$* **).

Suppose that $T^*(\mathfrak{A}^*)$ is inconsistent. Then for an open formula a of $T(\mathfrak{A})$ holds $a. -a \in Cn(\mathfrak{A}^*)$. On account of 5, $a. -a \in Cn_0(\mathfrak{A}^*)$.

Hence, by 1, $a. -a \in Cn_1(\mathfrak{A}^*)$. Consequently, by 2 there exist $\gamma_{\varrho_1}, \ldots, \gamma_{\varrho_r} \in \mathfrak{A}^*$ such that $\gamma_{\varrho_1} \cdot \ldots \cdot \gamma_{\varrho_r} \to a \cdot -a$ is provable in $T^*(\wedge)$. Let $\mathfrak{N}^*$ be a functionally free model of $T^*(\wedge)$, and let $\mathfrak{M}$ be the pseudomodel of $T(\mathfrak{A})$ such that $\mathfrak{N}^*$ is an extension of $\mathfrak{M}$. It follows from 3 that in every pseudomodel $\mathfrak{M}^*$ of $T^*(\mathfrak{A}^*)$, being an extension of $\mathfrak{M}$, $(\mathfrak{M}^*)\Phi(\gamma_{\varrho_1} \ldots \gamma_{\varrho_r} \to a. - -a) = 1$. Consequently $(\mathfrak{M}^*)\Psi(\gamma_{\varrho_1} \ldots \gamma_{\varrho_r}) = 0$. On account of 7, $(\mathfrak{M})\Phi(a_{\varrho_1} \ldots a_{\varrho_r}) = 0$. That is $(\mathfrak{M})\Psi(a_{\varrho_1} \ldots a_{\varrho_r} \to a. -a) = 1$. Since $a_{\varrho_1}, \ldots, a_{\varrho_r}, a$ are formulas of $T(\mathfrak{A})$, we have $(\mathfrak{N}^*)\Phi(a_{\varrho_1} \ldots a_{\varrho_r} \to a. -a) = 1$. $\mathfrak{N}^*$ being the functionally free model, we infer that $a_{\varrho_1} \cdot \ldots \cdot a_{\varrho_r} \to a. -a$ is provable in $T^*(\wedge)$, and by 6, in $T(\wedge)$ too. Consequently $a. -a \in Cn(\mathfrak{A})$.

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*) This theorem can be proved in a similar way for the non-classical theories considered in [A M], in particular, based on the *positive, minimal, intuitionistic* and *modal* logic.

**) This theorem can be analogously proved for some non-classical theories, e. g. for the theories based on *intuitionistic* logic. In this case, theorem 8 may also be obtained from an unpublished result of Łoś.

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ASTROPHYSICS

The Diffusion of Light through a Scattering Layer of Great Optical Thickness

by
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Communicated by H. NIEWODNICZAŃSKI at the meeting of March 28, 1955

The present note deals with a certain problem concerning the diffusion of light through a plane-parallel scattering layer of great optical thickness. A uniform layer of clouds illuminated on the upper boundary by parallel sun rays is an example of such a layer in Geophysics. Terrestrial clouds have very small absorption in the visual region of the spectrum; they represent an almost ideal case of pure scattering. Let τ — the optical thickness — vary from $\tau=0$ on the upper boundary, to $\tau=\tau_1$ on the lower. Exact formulae, giving for arbitrary τ_1 the intensity of transmitted and reflected radiation, are ill-adapted for numerical applications — particularly for great τ_1. Moreover, they do not give any possibility of getting a clear idea of the proportion in which for a given (great) τ_1 the light is divided into a reflected and transmitted part. In the present note an exact asymptotic formula is given for the total flux of the light transmitted through a thick scattering layer in the case of an isotropic or simple elongated scattering indicatrix. For the arbitrary shape of the indicatrix a general approximate solution is given with an arbitrary high degree of approximation.

Let the scattering indicatrix be given by the phase function p

$$p(\cos\theta)=\sum_{l=0}^{N}\tilde{\omega}_l P_l(\cos\theta), \tag{1}$$

where θ denotes the angle between the directions of the incident and scattered ray; $\tilde{\omega}_0, \tilde{\omega}_1, \ldots \tilde{\omega}_N$ are constants; P_l is the Legendre polynomial of degree l. For the case of pure scattering, $\tilde{\omega}_0=1$. Let, further, μ denote the cosine of the angle between the direction of the pencil of diffuse radiation and the outward normal to the layer, φ — the azimuth of this pencil. Let the direction of the parallel beam of (solar) rays illuminating the layer be specified by $-\mu_0, \varphi_0 (\mu_0 > 0)$; let the flux of this radiation per

unit area, normal to the direction of propagation, be on the upper boundary πF. If the intensity of the diffuse radiation travelling in the direction μ, φ at the optical depth τ be expanded in the form

$$I(\tau, \mu, \varphi) = \sum_{m=0}^{N}{}' I^{(m)}(\tau, \mu) \cos m(\varphi - \varphi_0),$$

we can see that the net flux of radiation $F(\tau)$ defined by the equation

$$F(\tau) = \int I \mu \, d\omega,$$

depends only on $I^{(0)}(\tau, \mu)$ ($d\omega$ is an element of solid angle and the integration is to be performed over the whole sphere). In the case of pure scattering ($\tilde{\omega}_0 = 1$) the problem admits the flux integral

$$F(\tau) = \pi F \mu_0 (e^{-\tau/\mu_0} + c),$$

where c is constant; determination of the flux of diffuse radiation transmitted through the layer is reduced in this way to determination of c.

In Chandrasekhar's method, employed by the present writer, we use, when solving the equation of radiative transfer, Gauss's quadrature formula for the integral figuring in this equation; we replace $I(\tau, \mu)$ (we omit the superscript $^{(0)}$) by $2n$ functions $I_i(\tau, \mu_i)$ taken for $2n$ directions μ_i $i = \pm 1, \ldots \pm n$; $\mu_{-i} = -\mu_i$ corresponding to zeros of the Legendre polynomial P_{2n}. This is the so-called n-th approximation. In the case when the phase function p is given by the simple expression $p = 1 - \tilde{\omega}_1 \cos\theta$ we have

$$(2) \qquad I_i = \tfrac{1}{4} F \left\{ \sum_{\alpha=-n+1}^{n-1}{}' \frac{L_\alpha e^{-k_\alpha \tau}}{1 + \mu_i k_\alpha} + L_0[(1 - \tfrac{1}{3}\tilde{\omega}_1)\tau + \mu_i] + L_n + e^{-\tau/\mu_i} \frac{H(\mu_0) H(-\mu_0)}{1 + \mu_i \mu_0} \right\};$$

$L_{\pm\alpha}$, $\alpha = 1, \ldots, n-1$; L_0, L_n are $2n$ constants of integration, which must be found from the boundary conditions stating that there is no diffuse radiation entering the upper ($\tau = 0$) boundary and no such radiation entering the lower ($\tau = \tau_1$) boundary

$$(3) \qquad I_{-i} = 0 \quad \text{for} \quad \tau = 0; \quad i = 1, \ldots, n;$$

$$(3 \text{ bis}) \qquad I_{+i} = 0 \quad \text{for} \quad \tau = \tau_1; \quad i = 1, \ldots, n;$$

$k_{\pm\alpha}$ ($\alpha = 1, \ldots, n-1$) are non-vanishing roots, which occur in pairs, of the characteristic equation

$$1 = \sum_{i=1}^{n}{}' \frac{a_i}{1 - \mu_i^2 k^2},$$

where a_i are weights appropriate for Gauss's quadrature formula and

$$H(\mu) = \frac{1}{\mu_1 \ldots \mu_n} \frac{(\mu + \mu_1) \ldots (\mu + \mu_n)}{(1 + k_1 \mu) \ldots (1 + k_{n-1} \mu)};$$

The formulae written above were given by Chandrasekhar [1].

Applying Gauss's quadrature formula for the determination of $F(\tau)$, we find

$$F(\tau)=\pi F\mu_0\left(e^{-\tau/\mu_1}+\frac{L_0}{3\mu_0}\right).$$

This formula holds good also in the case where the expansion of p is not limited to the two lowest terms. The flux of radiation emerging from the lower boundary is $F(\tau_1)$; to compute it we must find L_0. Determination of the constant L_0 from equations expressing the boundary conditions becomes simpler for great τ_1; we can omit then in these equations terms of the order $e^{-\tau_1}$ as compared with terms of the order e^0. In this way we find

$$L_0=\frac{W_{\mu_1}^{(n)}/W_1^{(n)}}{(1-\frac{1}{3}\tilde{\omega}_1)\tau_1+2W_{\mu}^{(n)}/W_1^{(n)}}, \tag{4}$$

where in the first row of the determinant $W_1^{(n)}$ there are $1's$ and in the row $\alpha+1$ $(\alpha=1,\ldots,n-1)$ $1/(1-\mu_i k_\alpha)$, $i=1,\ldots,n$; $W_{\mu}^{(n)}$ and $W_{\mu_1}^{(n)}$ are determinants which are obtained from $W_1^{(n)}$ by replacing the first row in it by

$$\frac{H(\mu_0)H(-\mu_0)}{1-\mu_1/\mu_0},\ldots,\frac{H(\mu_0)H(-\mu_0)}{1-\mu_n/\mu_0} \quad \text{or} \quad \mu_1,\ldots,\mu_n$$

respectively. When the expansion (1) of p is not curtailed at the second term the shape of formula (4) remains the same but in the determinants $W_1^{(n)}$, $W_{\mu}^{(n)}$, $W_{\mu_1}^{(n)}$ there appear now certain rational functions of k_α or of μ_0 which figure in the expression which in this case replaces (2). The essential conclusion from (4) is that for an arbitrarily high approximation and for an arbitrary shape of the scattering indicatrix the flux of diffuse radiation transmitted through a scattering layer of great optical thickness τ_1 depends on τ_1 and $\tilde{\omega}_1$ only through the product $(1-\frac{1}{3}\tilde{\omega}_1)\tau_1$ in the way disclosed by (4).

Returning to the case of the simple phase function $1+\tilde{\omega}_1\cos\theta$, we can significantly simplify expression (4) by passing from n to ∞. After some lengthy algebra we find that

$$W_{\mu}^{(n)}/W_1^{(n)}=\mu_1+\mu_2+\ldots+\mu_n-\left(\frac{1}{k_1}+\frac{1}{k_2}+\ldots+\frac{1}{k_{n-1}}\right);$$

this is ([1], p. 79) the n-th approximation of the limit $q(\infty)$ to which Hopf's function $q(\infty)$ tends in the transfer problem for stellar atmospheres in local thermodynamic equilibrium; $q(\infty)=0.710446\ldots$ Further it appears that

$$W_{\mu_1}^{(n)}/W_1^{(n)}=-\mu_0\frac{H(\mu_0)}{k_1\ldots k_{n-1}\mu_1\ldots\mu_n};$$

the product in the denominator of the right-hand side is equal $\frac{1}{\sqrt{3}}$ ([1], p. 74).

The exact – in the "infinite approximation" – shape of the function H, which characterizes the limb darkening in the photospheric problem, is given by several authors ([1] p. 125, last column of table XL) in tabular form. Finally we obtain instead of (4) an exact asymptotic formula

$$\frac{L_0}{3\mu_0} = -\frac{H(\mu_0)/\sqrt{3}}{(1-\frac{1}{3}\tilde{\omega}_1)\tau_1 + 2q(\infty)}. \tag{5}$$

When $e^{-\tau_1/\mu_0}$ can be omitted as compared with $L_0/(3\mu_0)$, formula (5) gives directly the fraction of light transmitted diffusely. For the single act of scattering, the net flux with respect to the plane normal to the direction of the incident beam depends only on $\tilde{\omega}_1$; we may therefore expect that formula (5) will give a good approximation also in the case, where the expansion of p is not limited to the two lowest terms only. Some computations made by the writer with the aid of the generalized formula (4) confirm this expectation.

For terrestrial clouds, formula (5) shows in a clear way the dependence of the illumination on $\tilde{\omega}_1$ and τ_1 and can serve as one of the bases for the determination of these parameters by observation. For the average size of water drops in a cloud, $\frac{1}{3}\tilde{\omega}_1$ approaches 1; the smallness of the coefficient of τ_1 in (5) and (4) explains the relative smallness of the variations in illumination by a cloudy sky, in spite of very large variations in the thickness of the clouds; this smallness explains also why, in spite of its enormous surface brightness, the disc of the sun is invisible through a cloud transmitting a considerable fraction (*e.g.* 20%) of incident light.

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On the Proofs of "Backward" Uniqueness for Some Non-Conservative Fields Describable by Differential Equations of the Hyperbolic Type

by
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1. The proofs of uniqueness for linear differential equations of the hyperbolic type are based on the assumption of continuity of the physical phenomena described by these equations (e. g. law of conservation of energy, equation of continuity for the probability current). Theorems on the uniqueness of the solution (uniqueness meaning that there exists at most one solution), as usually formulated for hyperbolic type equations or systems thereof, besides their mathematical value, have also a certain physical interpretation. Namely, if the initial conditions are prescribed within a bounded region, then the uniqueness of the solution of Cauchy's problem means that the velocity of propagation of "signals" is finite.

Theorems on the uniqueness of the solution of the problem of initial value were first formulated by Zaremba [1] and Rubinowicz [2], [3]. Their ideas were developed by other authors [4], [5]. Plebański [6] has recently formulated two theorems on uniqueness, valid for a wide class of differential equations in mathematical physics. These theorems will be briefly described in section 3. In the present paper, the results obtained by Plebański have been extended to some equations, such that the equation of continuity $I^{\nu}_{,\nu} + Q = 0$ follows from them as a consequence (see section 4). This makes possible the proof of "backward uniqueness" for Maxwell's macroscopic equations and for the equations of the scalar fields (see section 5).

2. In the present paper, the usual notations of the theory of relativity will be used, repeated upper and lower index meaning a sum from 0 to 3 (Greek indices) or from 1 to 3 (Latin indices), x^0 — time co-ordinate; x^1, x^2, x^3 — space co-ordinates. $\varphi_{,\alpha}$ designates $\frac{\partial \varphi}{\partial x^\alpha}$ and the symbol $d_k\tau$ in the integrand denotes integration over a k-dimensional continuum.

FA denotes the boundary of the region A. We shall deal with systems of linear partial differential equations ("the field equations") such that the characteristic quadratic form

$$A[\vec{\xi}] = a^{\alpha\beta}\xi_\alpha\xi_\beta, \tag{1}$$

where $a^{\alpha\beta} = a^{\alpha\beta}(x^\nu)$ satisfies, in the region under consideration, the following conditions:

$$a^{00} > 0, \tag{2.1}$$

$$a^{0k} = 0, \tag{2.2}$$

$$a^{\alpha\beta} = a^{\beta\alpha}, \tag{2.3}$$

$$a^{kl}\xi_k\xi_l < 0, \quad \text{if} \quad \vec{\xi} \neq 0. \tag{2.4}$$

Let R denote a bounded region of the three-dimensional continuum. We assume for simplicity that R lies on a hyperplane $x^0 = 0$. Let us consider a set of characteristic cones (we understand the characteristic cone as a surface formed by a set of bicharacteristics passing through a definite point) with vertices on a two-dimensional surface FR. We denote by Ω the set of such points x^ν of a four-dimensional continuum, so that each of them can be connected with an arbitrary point in R by means of a line which will not cross any of the characteristic cones. The set of points belonging to Ω so that $0 < x^0 < x_k^0$ (or $x_k^0 < x^0 < 0$, if $x_k^0 < 0$), is denoted by $\Omega(x_k^0)$. $R(x_k^0)$ is the common part of Ω and the hyperplane $x^0 = x_k^0$. Further: $T(x_k^0) = F\Omega(x_k^0) - R - R(x_k^0)$. We chose the direction of the normal unit vector $\vec{n}$, so as to have $n_0 > 0$.

3. The following lemma was shown by Plebański: if I^ν is a vector field such that

$$I^0 \geqslant 0, \quad a_{\alpha\beta}I^\alpha I^\beta \geqslant 0; \quad \text{where} \quad (a_{\alpha\beta}) = (a^{\alpha\beta})^{-1}, \tag{3}$$

then

$$\lambda = I^\alpha n_\alpha \geqslant 0 \quad \text{on} \quad R = R(0),\ R(x_k^0),\ T(x_k^0). \tag{4}$$

If an equation of continuity

$$I^\alpha{}_{,\alpha} + Q = 0 \tag{5}$$

follows from the equations of the field, then integrating both sides of (5) over $\Omega(x_k^0)$ we obtain:

$$\varepsilon \int_{\Omega(x_k^0)} Q\, d_4\tau - \int_{R(0)} I^0 d_3\tau + \int_{R(x_k^0)} I^0 d_3\tau + \int_{T(x_k^0)} \lambda\, d_3\tau = 0, \tag{6}$$

where $\varepsilon = \operatorname{sgn} x_k^0$. If $I^0|_R = 0$, we obtain from (6) with respect to (4):

$$I^0 = 0 \quad \text{in} \quad R(x_k^0), \quad \text{if} \quad Q \geqslant 0, \quad x_k^0 > 0,$$

$$I^0 = 0 \quad \text{in} \quad R(x_k^0), \quad \text{if} \quad Q \leqslant 0, \quad x_k^0 < 0.$$

If vanishing of I^0 is equivalent to vanishing of the corresponding functions

of the field, then the theorem on uniqueness follows from the above two lemmata in the known way. In the case of $Q=0$ ("conservative field") the uniqueness of the solution can be proved in the whole region Ω, and in particular, in Ω^- (i. e. in $\Omega(x_k^0)$ for any $x_k^0<0$) which is the case of "backward" uniqueness.

4. The above theorems do not determine the uniqueness if the sign of Q is not constant; they also give no information concerning the behaviour of the field in $\Omega(x_k^0)$ for $x_k^0>0$ provided $I^0|_R=0$ in the case of $Q\leqslant 0$, and they do not determine the "backward" uniqueness in the case of $Q\geqslant 0$, which is the case of Maxwell's equations for conducting media. We shall now prove a theorem from which the "backward" uniqueness for Maxwell's macroscopic equations follows as one of the consequences.

THEOREM. *If a vector field I^ν, continuous in Ω, satisfies the conditions* (3) *and* (5) *and there exists such a constant M, that for each $x^\nu\in\Omega$ there is*

$$|Q|\leqslant MI^0, \tag{7}$$

then the vanishing of I^0 on R implies the vanishing of I^0 in Ω.

We shall prove this theorem for $x_k^0>0$ and $x_k^0<0$ simultaneously. From (6) we get

$$0\leqslant E(x_k^0)=\int\limits_{R(x_k^0)} I^0 d_3\tau=\int\limits_{R(0)} I^0 d_3\tau-\varepsilon\int\limits_{\Omega(x_k^0)} Q\,d_4\tau-\int\limits_{T(x_k^0)}\lambda d_3\tau. \tag{8}$$

Taking into account (4) we obtain

$$0\leqslant E(x_k^0)\leqslant -\varepsilon\int\limits_{\Omega(x_k^0)} Q\,d_4\tau+\int\limits_{R(0)} I^0 d_3\tau\leqslant\int\limits_{\Omega(x_k^0)}|Q|\,d_4\tau+E(0), \tag{9}$$

hence by (7):

$$0\leqslant E(x_k^0)\leqslant E(0)+\int\limits_{\Omega(x_k^0)} MI^0 d_4\tau=E(0)+\varepsilon M\int\limits_0^{x_k^0} E(x^0)\,dx^0. \tag{10}$$

Denoting

$$\psi(x_k^0)=e^{-\varepsilon M x_k^0}\int\limits_0^{x_k^0} E(x^0)\,dx^0$$

we obtain:

$$\psi'(x_k^0)=-\varepsilon M\psi+e^{-\varepsilon M x_k^0}E(x_k^0).$$

Thus (10) becomes

$$\psi'(x_k^0)\leqslant E(0)\,e^{-\varepsilon M x_k^0}. \tag{11}$$

Now let x_n^0 be an arbitrary number so that $\operatorname{sgn} x_n^0=\operatorname{sgn} x_k^0$. Integrating

both sides of inequality (11) over the interval $(0, x_n^0)$, if $x_n^0 > 0$, or over $(x_n^0, 0)$ if $x_n^0 < 0$, we obtain:

$$\varepsilon[\psi(x_n^0) - \psi(0)] \leqslant -\frac{E(0)}{M}(e^{-\varepsilon M x_n^0} - 1)$$

and, since $\psi(0) = 0$, we get:

$$\varepsilon \int_0^{x_n^0} E(x^0)\,dx^0 \leqslant \frac{E(0)}{M}(e^{\varepsilon M x_n^0} - 1). \tag{12}$$

According to (12), and bearing in mind that $\varepsilon x_n^0 = |x_n^0|$, we obtain from (10) for an arbitrary x_k^0:

$$0 \leqslant \int_{R(x_k^0)} I^0 d_3\tau \leqslant E(0) + E(0)(e^{\varepsilon M x_k^0} - 1) = e^{M|x_k^0|} \int_R I^0 d_3\tau. \tag{13}$$

Putting $I^0|_R = 0$ into (13), we obtain proof of our theorem.

If I^0 is interpreted as the density of the field energy, then (13) states that the changes of energy in Ω are at most of an exponential nature.

Conclusion: if

1^0 the functions $\varphi^A (A = 1, 2, \ldots, N)$ satisfy linear differential equations with the characteristic quadratic form $A[\vec{\xi}] = a^{\alpha\beta}\xi_\alpha \xi_\beta$,

2^0 the form $A[\vec{\xi}]$ has the properties (2.1)–(2.4),

3^0 the field equations lead to the equation of continuity (5),

4^0 the vector field I^ν is continuous and satisfies (3),

5^0 there exists a constant M satisfying (7) inside Ω,

6^0 vanishing of I^0 is equivalent to vanishing of φ^A,

there exists at most one solution of the field equations in Ω satisfying on R the prescribed initial conditions.

Proof. If two solutions φ_1^A, φ_2^A existed, then in view of the linearity of the equations, $\varphi_*^A = \varphi_1^A - \varphi_2^A$ would also be a solution satisfying the initial homogeneous conditions $\varphi_*^A|_R = 0$. However, the vector I_*^ν corresponding to the field φ_*^A would vanish in R, and thus, by virtue of the theorem which has been proved, it would also vanish in Ω. Thus it follows that $\varphi_*^A \equiv 0$ i. e. $\varphi_1^A = \varphi_2^A$ inside Ω.

5. Examples.

a. Maxwell's equations.

From the macroscopic equations of the electromagnetic field we can obtain a conservation principle of the form

$$\operatorname{div} \vec{S} + \frac{\partial W}{\partial t} = -\sigma E^2 = -Q, \tag{14}$$

where $\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{H}$, $W = \frac{1}{8\pi}(\varepsilon E^2 + \mu H^2)$; ε, μ, σ are positive functions of

the point x^k, and $\varepsilon \geqslant 1$. The components of $\vec{E}$ and $\vec{H}$ satisfy the wave equation with the characteristic form $\frac{\varepsilon\mu}{c^2}\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2$ which is of type (2). Denoting $(x^0, x^1, x^2, x^3) = (t, x, y, z)$, $(I^0, I^1, I^2, I^3) = (W, \vec{S})$ we can write (14) in the form of (5). It can easily be proved that vector I^α satisfies the conditions (3). Assuming $M = 8\pi \cdot \sup \sigma$, we find (in view of $\varepsilon \geqslant 1$) that condition (7) is satisfied. Assumptions $1^0 - 6^0$ are thus fulfilled and it follows that there exists at most one electromagnetic field in Ω taking prescribed values in a bounded region R.

b. Scalar field equation.

Let us consider the equation

$$K[u] = (a^{\alpha\beta} u_{,\alpha})_{,\beta} + b^\alpha u_\alpha + cu = 0, \tag{15}$$

where

$$\begin{cases} a^{\alpha\beta} & \text{are differentiable functions satisfying (2),} \\ c \geqslant 0 & \text{inside } \Omega, \\ b^\alpha & \text{are functions continuous in } \Omega. \end{cases} \tag{16}$$

If $b^\alpha = 0$, (15) is the Euler-Lagrange equation of the following problem:

$$\delta \int_{\Omega} L \, d_4\tau = 0, \qquad \text{where} \quad L = a^{\alpha\beta} u_{,\alpha} u_{,\beta} - cu^2.$$

The impulse-energy tensor

$$T^\beta_\alpha = \delta^\beta_\alpha L - u_{,\alpha} \frac{\partial L}{\partial u_{,\beta}} = \delta^\beta_\alpha L - 2 a^{\mu\beta} u_{,\mu} u_{,\alpha}$$

satisfies the equation

$$T^\beta_{\alpha,\beta} = 2 u_{,\alpha} b^\nu u_{,\nu}.$$

Defining $I^\beta = -T^\beta_0$, we obtain: $I^\beta_{,\beta} = -2 u_{,0} b^\nu u_{,\nu} = -Q$ and

$$I^0 = a^{00} u^2_{,0} - a^{mn} u_{,m} u_n + cu^2 \geqslant 0. \tag{17}$$

If, further, $(a_{\alpha\beta}) = (a^{\alpha\beta})^{-1}$, then

$$a_{\alpha\beta} I^\alpha I^\beta = a_{00} L^2 + 4 cu^2 u^2_{,0} \geqslant 0. \tag{18}$$

Denoting the upper bound of the function $2|b^0| + |b^1| + |b^2| + |b^3|$ in $\bar{\Omega}$ by M_1 we get:

$$|Q| = |2 u_{,0} b^\nu u_{,\nu}| \leqslant M_1 \sum_{\alpha=0}^{3} u^2_{,\alpha}. \tag{19}$$

Let $\varkappa$ denote the smallest lower bound of the moduli of characteristic roots of the form $A[\vec{\xi}]$. In view of (2.1) and (2.4), it must be that $\varkappa > 0$, and

$$a^{00} \xi_0^2 - a^{mn} \xi_m \xi_n \geqslant \varkappa \sum_{\alpha=0}^{3} \xi_\alpha^2. \tag{20}$$

Taking into account (17), (19) and (20), we get:

$$|Q| \leqslant \frac{M_1}{\varkappa}\,(a^{00}u^2{}_{,0} - a^{mn}u_{,m}u_{,n}) \leqslant \frac{M_1}{\varkappa}\,I^0 = MI^0. \tag{21}$$

It can be seen from equations (16)–(21) that in the case of equation (15) the assumptions 1^0-5^0 of the conclusion on uniqueness are satisfied. Further, vanishing of I^0 is equivalent to vanishing of $u_{,n}$. Thus we easily obtain a theorem on the uniqueness of the solutions of the equation: $K[u]=f(x^r)$, with Cauchy-type conditions for u prescribed on R.

c. It should be noted that the third of the assumptions (16), which makes estimation (19) possible, is of essential importance; e. g. the equation

$$u_{tt} - \Delta u - \frac{u}{\sin t} + u = 0$$

has a non-trivial solution $u = \cos t - 1$, vanishing together with its partial derivatives on the hyperplane $t=0$. This example shows also that the assumption $|Q| \leqslant MI^0$ is essential for the truth of the theorem.

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THEORETICAL PHYSICS

Note on Coulomb Effects in Stripping Reactions

by

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Coulomb corrections to the Butler cross-section for the (d, p) and $(d, n$ reactions were recently [1] discussed with the help of the "zero-range" Horowitz-Messiah approximation for the $n-p$ interaction. The integrations in the matrix elements were performed with the help of complicated numerical computations.

It appears that on applying the H.-M. approximation and the asymptotical exponential form of the wave function of the captured particle in the particular case of the S-state in the one-particle model, one can calculate analytically the Born approximation matrix element for the (d, p) or (d, n) reaction with the plane waves replaced by the Coulomb wave functions.

Let us consider first the (d, p) reaction. The usual Born approximation c. m. differential cross-section may be written in the form:

$$(1) \qquad \sigma(\theta)=\frac{M_d^* M_p^*}{(2\pi\hbar^2)^2}\frac{k_p}{k_d}\frac{1}{3(2I_i+1)}\sum_{\substack{\mu_p\mu_d\\ \mu_i\mu_f}}{}' \int d\sigma_n\, d\sigma_p\, d\xi\, d_{(3)}\bar{r}_n\, \chi_f^*(\xi, \bar{r}_n\, \sigma_n)\, \chi_i(\xi) e^{i\bar{k}_d \bar{r}_n} \cdot$$

$$\cdot e^{-i\bar{k}_p' \bar{r}_n} \cdot \int d_{(3)}\bar{\varrho}\, e^{-i\bar{k}_\varrho} V_{np}(\varrho)\psi(\varrho) S_{\mu_d}(\sigma_n, \sigma_p)\Big|^2,$$

where M_d^*, M_p^* are the deuteron and proton reduced masses; k_d, k_p their respective momenta; μ_p, μ_d, μ_i, μ_f — magnetic numbers; I_i — total angular momentum quantum number of the initial nucleus; σ_n, σ_p — spin variables of the neutron and the proton; $\bar{r}_n$ — vectors from the initial nucleus (i) to the neutron and $\bar{\varrho}$ — the vector from the neutron to the proton; $\bar{k}_p' = \frac{M_i}{M_f}\bar{k}_p$ and $\bar{K} = \frac{\bar{k}_d}{2} - \bar{k}_p$; χ_i and χ_f are the internal wave functions of the initial and final nuclei; $V_{np}(\varrho)$ is the n-p interaction and $\psi(\varrho)\cdot S_{\mu_d}(\sigma_n, \sigma_p)$ — the deuteron internal wave function.

On assuming the Mayer shell (one-particle) model we may write: $\chi_f(\xi, \bar{r}_n, \sigma_n) = \chi_i(\xi)\Phi(\bar{r}_n, \sigma_n)$; if the neutron is captured to the S-state we have: $\Phi(\bar{r}_n, \sigma_n) = \Phi(r_n)\chi_{\mu_n}(\sigma_n)$. In this non-Coulomb case we obtain:

$$(2) \quad \sigma(\theta) = \frac{M_d^* M_p^* k_p}{(2\pi\hbar^2)^2 k_d} \cdot S^2 \left| \int d_{(3)}\bar{r}_n \Phi(r_n) e^{i\bar{k}_d \bar{r}_n} \cdot e^{-i\bar{k}_p' \bar{r}_n} \cdot \int d_{(3)}\bar{\varrho} e^{-i\bar{K}_\varrho} V_{np}(\varrho)\psi(\varrho) \right|^2,$$

where the constant factor

$$S^2 = \frac{1}{3} \sum_{\mu_p, \mu_d, \mu_n} \left| \int d\sigma_n d\sigma_p \chi_{\mu_n}^*(\sigma_n) \chi_{\mu_p}^*(\sigma_p) S_{\mu_d}(\sigma_n \sigma_p) \right|^2$$

(in our case $I_i = 0$).

On applying the H.-M. approximation we obtain a constant value for the integral over ϱ. The only remaining integral is:

$$(3a) \quad I_\varkappa = \int d_{(3)}\bar{r}\Phi(r) e^{i\bar{k}_d \bar{r}} \cdot e^{i\bar{k}_p' \bar{r}},$$

On replacing the plane waves by the respective Coulomb wave functions, we obtain:

$$(3b) \quad I_\varkappa^c = \int d_{(3)}\bar{r}\Phi(r) F_d(\bar{r}) F_p^*(\bar{r}),$$

where:

$$F_d(\bar{r}) = e^{-\frac{\pi n_d}{2}} \Gamma(1 + i n_d) e^{ik_d r} {}_1F(-i n_d, 1;\ i(k_d r - \bar{k}_d \bar{r}))$$

$$F_p^*(\bar{r}) = e^{-\frac{\pi n_p}{2}} \Gamma(1 + i n_p) e^{-ik_p' r} {}_1F(-i n_p, 1;\ i(k_p' r + \bar{k}_p' \bar{r}))$$

$$n_d = \frac{Ze^2 M_d^*}{\hbar^2 k_d}, \quad n_p = \frac{Ze^2 M_p^*}{\hbar^2 k_p}$$

${}_1F$ is the confluent hypergeometrical function.

On assuming

$$\Phi(r) = \sqrt{\frac{\gamma}{2\pi}}\, e^{-\gamma r}, \qquad \gamma = \sqrt{\frac{2\mu_n \epsilon_n}{\hbar^2}}$$

(ϵ_n — binding energy of the captured neutron),
we can calculate $I_\varkappa^c$ analytically by the Sommerfeld method [2]:

$$I_\varkappa^c = I_\varkappa e^{-\frac{\pi}{2}(n_p + n_d)} \Gamma(1 + i n_p)\, \Gamma(1 + i n_d) \left\{ \frac{k_d^2 - k_p'^2 + \gamma^2 - i2\gamma k_p'}{\varkappa^2 + \gamma^2} \right\}^{i n_p} \cdot \left\{ \frac{k_p'^2 - k_d^2 + \gamma^2 - i2\gamma k_d}{\varkappa^2 + \gamma^2} \right\}^{i n_d} \cdot {}_2F_1\left(-i n_d, -i n_p, 1; \lambda \sin^2 \frac{\theta}{2} \right),$$

where $\bar{\varkappa} = \bar{k}_d - \bar{k}_p'$, ${}_2F_1$ is the hypergeometrical function, λ is a real number < 0, which it is convenient to represent in the form:

$$\lambda = \frac{4 k_d k'_p [(k_d + k'_p)^2 - \gamma^2 + i 2\gamma (k'_p + k_d)]}{(k'^2_p - k^2_d + \gamma^2 - i 2\gamma k_d)\ (k^2_d - k'^2_p + \gamma^2 - i 2\gamma k'_p)} \qquad \text{and} \qquad \theta = \sphericalangle (\bar{k}_d, k'_p)$$

(the scattering angle).

The ratio of the cross-section resulting from (3b) to $\sigma(\theta)$ is:

$$(4) \qquad \frac{\sigma^c(\theta)}{\sigma(\theta)} = e^{-\pi(n_p + n_d)} \frac{\pi n_p}{sh \pi n_p} \frac{\pi n_d}{sh \pi n_d} e^{n_p \operatorname{arctg} \frac{2\gamma k'_p}{\gamma^2 + k^2_d - k'^2_p}}$$

$$\cdot e^{n_d \operatorname{arctg} \frac{2\gamma k_d}{\gamma^2 + k'^2_p - k^2_d}} \left| {}_2F_1\left(-i n_d, -i n_p, 1; \lambda \sin^2 \frac{\theta}{2} \right) \right|^2$$

Equation (4) gives a flattening of the angular distribution at larger angles and a very considerable decrease in the absolute values of the cross-section sensitive to Z and to the velocities of the deuteron and the proton.

As an illustration let us consider the reaction $^{16}O(d, p)\,^{17}O^*$), when the first excited state is formed (S-state, Q-value 1.04 MeV). The corresponding angular distributions $\sigma(\theta)$ and $\sigma^c(\theta)$ for smaller angles are indicated in Fig. 1 for the incident deuteron energy 4.5 MeV. $\sigma^c(\theta = 0)/\sigma(\theta = 0) \cong 0.02$ for 4.5 MeV deuterons. The calculated $\sigma^c(\theta = 0) = \cong 13.1 \frac{mb}{\text{ster}}$ is evidently too small in view of the experimental value obtained by Burge et al. [3] for $\theta = 15^0$ and 7.73 MeV deuterons $\sigma_{\text{exp}}(\theta = 15^0) \sim 50 \frac{mb}{\text{ster}}$ and especially in view of the theoretically predicted reduction of the cross-section so as to include an interaction between the proton and the bombarded nucleus [4]. It seems that this discrepancy is mostly connected with neglect of the important (especially for smaller energies) probability of a compound nucleus formation and also with neglect of the Oppenheimer-Phillips process (i. e. the Coulomb deformation of the deuteron function $\psi(\varrho)$). In any case, it appears to be certain that it is impossible to make a reasonable comparison between the absolute values of $\sigma(\theta)$ and the values obtained experimentally, without taking into consideration the Coulomb effect.

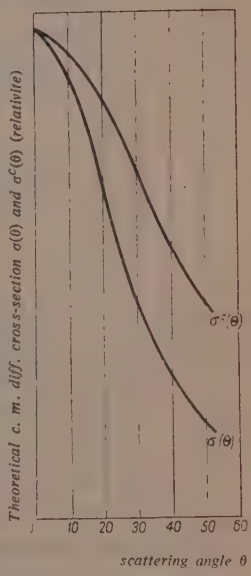

Fig. 1. Theoretical angular distribution of protons from $^{16}O(d, p)^{17}O^*$ for incident deuteron energy 4.5 MeV: $\sigma^c(\theta)$ — Coulomb effect included, $\sigma(\theta)$ — no Coulomb effect

On neglecting the Coulomb deformation of the internal wave function $\Phi(r)$ or the captured proton in the case of the (d, n) reaction, we would have for $\sigma^c(\theta)/\sigma(\theta)$ eq. (4) with $n_p = 0$; $\sigma^c(\theta)/\sigma(\theta)$ would be isotropic and the absolute values would be considerably greater (though of the same order) than for the (d, p) reaction with the neglect of the O.P. process.

Further, the (d, n) processes arising from the compound nucleus formation are more probable than the corresponding (d, p) processes. The resultant probabilities of all the discussed processes would change the ratio $\sigma^c_{dp}/\sigma^c_{dp}$.

A somewhat similar situation is seen in the case of the $^6\mathrm{Li}(n, \alpha)^3\mathrm{H}$ reaction. This reaction was recently [5] investigated from the point of view of the "α – particle + deuteron" model with the assumption that the "$\alpha + d$" system is in the S state.

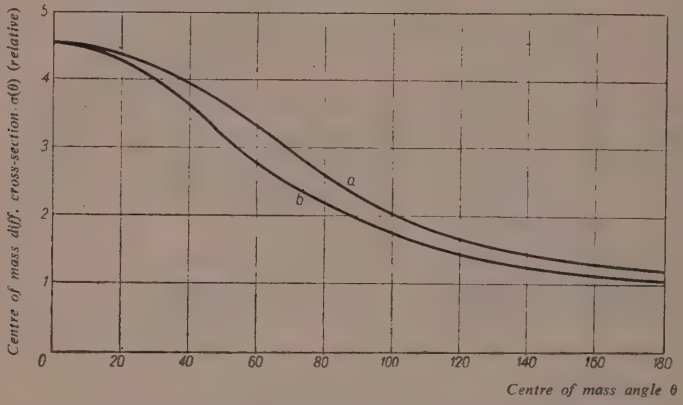

Fig. 2. Angular distribution of tritons from $^6\mathrm{Li}$ $(n, \alpha)^3$ H for incident neutron energy 1.5 MeV: a — experimental (Weddel and Roberts [12]), b — theoretical.

The Born approximation differential cross-section in the c. m. system for this reaction was obtained in [5] in the form:

$$\sigma(\theta)=\frac{M_t^* M_{n_0}^*}{(2\pi\hbar^2)^2}\frac{k_t}{k_{n_0}}\left[\frac{1}{2\cdot 3}\sum_{\mu_0,\mu_d,\mu_t}|\langle\chi_\alpha\chi_t e^{i\bar{k}_t\left(\bar{r}-\frac{\bar{s}}{3}\right)}|V_{n_0 d}|\psi_0\rangle|^2\right], \tag{5}$$

where: $M_t^*=12\ 7M$, $M_{n_0}^*=6/7M$ are respectively the triton and the neutron reduced masses; $\bar{k}_t$ and $\bar{k}_{n_0}$ are the corresponding momenta in the c. m. system; μ_i are the resp. magnetic quantum numbers, $\chi_\alpha=\chi_\alpha(\xi)$ is the internal wave function of the α-particle, $\chi_t=S_{\mu_t}(\sigma_0, \sigma_n, \sigma_p)\cdot R_t(s,u)$ – of the triton, σ_i being the corresponding spin variables; $\bar{s}=\frac{\bar{r}_n+\bar{r}_p}{2}-\bar{r}_{n_0}$, $\bar{u}=\frac{\sqrt{3}}{2}(\bar{r}_n-\bar{r}_p)$ are the internal co-ordinates of the triton; $\bar{r}$ – is the vector from the α-particle to the deuteron; $V_{n_0 d}$ is the incident neutron-deuteron interaction. ψ_0 is defined by

$$\psi_0=\chi_{\mu_0}(\sigma_0)e^{i\bar{k}n_0\left(\frac{2}{3}\bar{r}-\bar{s}\right)}\chi_\alpha(\xi)S_{\mu_d}(\sigma_n\sigma_p)\psi\left(\frac{2}{\sqrt{3}}u\right)\Phi(r).$$

$\psi(\varrho)$ is the unperturbed deuteron internal function and $\Phi(r)$ is the S-state "$\alpha + d$" system internal function.

Now we shall not apply the transformation of the matrix element as was done in [5], but the "zero-range" interaction

$$V_{n_0 d} = -\frac{3\pi a_{nd}\hbar^2}{M}\,\delta(\bar{s}), \quad \text{where} \quad a_{nd} = \sqrt{\frac{2}{3}a_4^2 + \frac{1}{3}a_2^2} \approx 5.08\cdot 10^{-13}\,\text{cm},$$

the n-d "scattering length" given by Christian and Gammel [7], as was done in [6] *).

As a result we have:

$$\frac{1}{2\cdot 3}\sum{}' |< \; >|^2 = \frac{1}{4}\left(\frac{3\pi a_{nd}\hbar^2}{M}\right)^2 |b_k|^2 \cdot |I_t|^2,$$

where

$$b_k = \int R_t(u,0)\,\psi\left(\frac{2}{3}u\right) d_{(3)}\bar{u} \quad \text{and} \quad I_t = \int \Phi(r)\, e^{i\bar{K}\bar{r}}\, d_{(3)}\bar{r},$$

where

$$\bar{K} = \tfrac{2}{3}\,\bar{k}_{n_0} - \bar{k}_t.$$

Since the Coulomb interaction of the "$\alpha + d$" system is small (see e. g. [8]), the Coulomb deformation of $\Phi(r)$ is neglected **) Similarly, as in [9], we shall apply for $\Phi(r)$ the asymptotical form $\sqrt{\frac{\bar{\alpha}}{2\pi}}\,\frac{e^{-\alpha r}}{r}$, where $\alpha = \sqrt{\frac{\frac{8}{3}M\epsilon}{\hbar^2}}$ ($\epsilon = 1.477$ MeV = the binding energy of the deuteron in the ^{6}Li nucleus).

Let us now discuss the Coulomb effect of the outgoing triton wave in (5). The Coulomb polarization of the strongly bound structure of the triton is neglected.

On putting the Coulomb wave function in place of the triton plane wave in (5), we obtain, instead of I_t:

$$I_t^c = \int \Phi(r)\, e^{i\frac{2}{3}\bar{k}_{n_0}\bar{r}}\, F_t^*(r)\, d_{(3)}\bar{r}, \tag{6}$$

*) The "zero-range" approximation is reasonable for neutron energies considerably less than 10 MeV; here it is equivalent mathematically to the H.-M. approximation.

**) $\Phi_c(r) = \frac{A}{r} W_{-n,\frac{1}{2}}(2\alpha r)$, where $n = \frac{2e^2 M_d^*}{\hbar^2 \alpha}$ and $W_{-n,\frac{1}{2}}$ is the Whittaker function, would be the corresponding function to $\Phi(r) = \Phi_\alpha(r) = \sqrt{\frac{\alpha}{2\pi}}\,\frac{e^{-\alpha r}}{r}$ if the internal Coulomb effect of the "$\alpha + d$" system were taken into consideration. Similarly as in [1] (Appendix 2), $\Phi_c(r)$ may be approximated in the important region of r by some $\Phi_{\alpha'}(r)$. The value $\alpha' = 3.34\cdot 10^{12}\,\text{cm}^{-1}$ may be calculated according to the method used in [1], if $\alpha' = 3.08\cdot 10^{12}\,\text{cm}^{-1}$. The small difference between α' and α has little effect on $\sigma(\theta)$ and $\sigma^c(\theta)$.

where:

$$F_l^*(\bar{r}) = e^{-\frac{\pi n_l}{2}} \Gamma(1+in_l) e^{-i\bar{k}_l\bar{r}} {}_1F(-in_l, 1; i(k_l r + \bar{k}\bar{r})), \qquad n_l = \frac{2e^2 M_l^*}{\hbar^2 k};$$

The last integral is:

$$I_l^c = e^{-\frac{\pi n_l}{2}} \Gamma(1+in_l) \cdot I_l \cdot \left\{ \frac{\frac{4}{9}k_{n_0}^2 - k_l^2 + \bar{\alpha}^2 - i2\bar{\alpha}k_l}{K^2 + \bar{\alpha}^2} \right\}^{in_l}$$

and the ratio of the cross-section resulting from (3) to $\sigma(\theta)$:

$$\frac{\sigma^c(\theta)}{\sigma(\theta)} = \frac{|I_l^c|^2}{|I_l|^2} = e^{-\pi n_l} \frac{\pi n_l}{sh\pi n_l} e^{-2n_l \operatorname{arc\,tg} \frac{2\bar{\alpha}k_l}{k_l^2 - \frac{1}{3}k_{n_0}^2 - \bar{\alpha}^2}}. \tag{7}$$

The angular distribution remains unchanged. However, as can be seen from Fig. 1, the agreement with experimental results of the angular distribution for 1.5 MeV neutrons is quite good (considerably better than that in [5]).

We can conclude then that the "zero-range" interaction, wiping out the factor I_h from [5], is even a better approximation for small energies.

Similarly as in [1] the comparison between the absolute values of $\sigma(\theta=0)$ and $\sigma^c(\theta=0)$ is more important. The values of these quantities for 1.5 MeV neutron energy are shown in Table I for 3 cases:

1) R_l – the Gaussian function used in [10] and ψ - the usual square well deuteron function,

2) R_l – the same as in 1) and ψ – the Hulthén function with the parameter values given by Chew and Goldberger [11],

3) R_l – the Irving function defined in [12] and ψ – the same as in 2).

TABLE I

	1	2	3
$\sigma(\theta=0)$	215.3	222.1	196.1
$\sigma^c(\theta=0)$	94.7	97.7	86.3

$\sigma(\theta=0)$ and $\sigma^c(\theta=0)$ in $\frac{mb}{\text{ster}}$ for 1.5 MeV neutrons.

The ratio σ^c/σ is equal, for neutron energies of 1.5, 2 and 14 MeV, to 0.44; 0.43 and 0.62 respectively. The use of $\Phi(r) = \sqrt{\frac{\alpha}{2\pi}} \frac{e^{-\alpha r}}{r}$ gives too great $\sigma(0)$. In [5] the value $\sigma(0) = 72 \frac{mb}{\text{ster}}$ for 1.5 MeV neutrons was obtained for the best for small energies "$\alpha - d$" interaction radius $R_0 = 1.10^{-13}$ cm., R_l-Gaussian and ψ – Hulthén. Since the corresponding experimental

value [13] is $\sigma_{exp}(0) = 46 \frac{mb}{ster}$, on extrapolating our result $\sigma^c/\sigma \simeq 0.44$ to the approximation of [5], we would obtain a better agreement with experiment, for the absolute value.

Further, we see that σ^c/σ increases together with the neutron energy. In [5] too low values of $\sigma(0)$ were obtained for 14 MeV neutrons. A much better (greater) value of

$$[\sigma(\theta=0)]_{14}/[\sigma(\theta=0)]_{1,5}$$

is hoped for as a result of the Coulomb effect.

I am much indebted to Professor L. Infeld for his kind interest in this work and to J. Dąbrowski for many helpful discussions.

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BULLETIN DE L'ACADÉMIE
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EXPERIMENTAL PHYSICS

The Energetic Part of the γ-Ray Spectrum of ThD Determined with the Aid of a Diffusion Cloud Chamber

by

S. WIKTOR

Communicated by H. NIEWODNICZAŃSKI at the meeting of March 28, 1955

In the γ-ray spectrum accompanying the natural radioactivity of elements of the thorium series, the most energetic is commonly considered to be the line with an energy of 2.62 MeV, emitted by ThD ($^{208}_{82}$Pb). However, there are a few results of experimental investigations which indicate that in the natural radioactivity of thorium there exists a line of still greater energy, viz., 3.20 MeV. The investigations of Alichanov and his co-workers [1], [2], carried out with the aid of a magnetic β-ray spectrograph with pair electrons produced by γ-rays from a ThC source, point to the existence of the 3.20 MeV line. Its relative intensity with respect to the well-known strong line at 2.62 MeV was evaluated as being about 1:40. The nuclear emulsion method applied recently to this problem by Niewodniczański and Wielowiejska [3] confirmed the existence of the 3.20 MeV line. For its intensity relative to that with an energy of 2.62 MeV a value of about 1:10 was found. The purpose of the present work was to give further confirmation of the existence of the 3.20 MeV line in the γ-ray spectrum of ThD and to measure the relation of its intensity to that of the 2.62 MeV line with the use of a new method based on a diffusion cloud chamber.

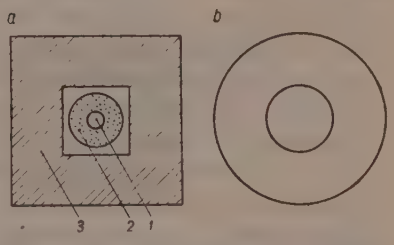

Fig. 1. a — neutron source, b — diffusion chamber; 1. MsTh. 2. Be. 3. Pb

A source of photo-neutrons (Fig. 1) was situated near a diffusion chamber which served for observing and photographing the tracks of recoil protons produced by collisions of neutrons with hydrogen nuclei inside the chamber. The chamber (Fig. 2) consisted of a glass cylinder,

25 cm. in diameter and 7 cm. in height. The walls of the cylinder were 8 mm. thick. The bottom of the chamber was made of a blackened aluminium plate, 6 mm. thick. The top cover, made of a brass plate 5 mm. thick, contained an externally heated methyl alcohol vapour source and was provided with a glass window for observation. The top cover together with the alcohol container were kept at a constant temperature, in the region of 20-25° C, by the flow of water from an ultra-thermostat through a spiral tube. The bottom of the chamber was cooled by solid carbon dioxide (so-called dry ice). The connections of the glass cylinder with metal plates were insulated with rubber.

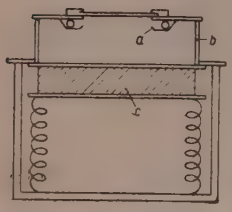

Fig. 2. *a* — vapour source, *b* — glass cylinder, *c* — dry ice.

In order to remove the excessive ions which were produced in the chamber in very great numbers by the strong γ-radiation from the source, a sweeping electrostatic field (voltage of about 500 V) was continuously maintained between the bottom and the cover. The sensitive region of the chamber occupied a volume extending from the bottom to a height of about 3 cm. The source of photoneutrons consisted of about 20 m Curie of MsTh, surrounded by a layer (2.8 cm. thick) of powdered beryllium metal. This source was shielded from the chamber by a 10 cm. layer of lead. This thickness of the Pb shield was found experimentally to be the most suitable for reducing the background of γ-radiation in the chamber while, at the same time, it did not diminish too much the intensity of the neutron beam.

The chamber, working under atmospheric pressure, was filled with air and an admixture of hydrogen which was admitted continuously into the chamber in a weak stream. In the sensitive region of the chamber there appeared in the course of an hour an average of about 250 tracks of recoil protons and of background α-rays produced from radioactive contaminations.

The energy of the recoil protons was determined by measuring the lengths of their tracks in the chamber. If the energy of the γ-ray photon amounts to 2.62 MeV, then taking into consideration the threshold energy for the reaction $^9\mathrm{Be}(\gamma, n)\,^8\mathrm{Be}$, amounting to 1.67 MeV, and the recoil energy of the ^{8}Be nucleus, amounting to 0.12 MeV, the liberated neutron should acquire a kinetic energy of 0.83 MeV. Similarly, the 3.20 MeV photon should transfer 1.36 MeV of kinetic energy to the liberated neutron. In the case of a head-on collision we can assume that the neutron transfers its total energy to the recoil hydrogen nucleus. In air, under normal conditions, the range of 0.83 MeV protons is, according to Livingston and Bethe [4], 17 mm., while that of 1.36 MeV protons is 37 mm. The 37 mm. tracks were distinctly visible among the many 17 mm. tracks directly observed and photographed.

Owing to the small efficiency of the neutron source and the limited time of exposure, the number of photographed tracks was too small to ensure a more exact estimation of the intensity of both γ-ray lines. The recoil proton tracks were then counted visually. During counting, the tracks were divided into two groups: those of the lst group having approximate length ranging from 5 to 20 mm., and those of the 2nd group from 20 to 40 mm. Tracks deviating from the direction of the neutron beam at an angle greater than 45°, were not counted. The second group contained also tracks produced by the α-rays from gaseous radioactive contaminations. Most of these tracks were produced by the α-rays of radon and possessed a maximum length slightly exceeding 10 mm. In order to distinguish these tracks from those due to proton recoils, the track counting was performed repeatedly under the same experimental conditions, only with the beryllium powder removed.

The total time taken to count the tracks in the chamber was 2445 minutes. During this time 6926 tracks were recorded. The results of the counting are given in the following table:

TABLE

	Total number of tracks observed		α-ray background		Difference (protons)	
		Mean number per hour		Mean number per hour		Mean number per hour
Duration of recording (min).	1305	—	1140	—	—	—
Number of tracks of the lst group	4704	216	95	5	4595	211
Number of tracks of the 2nd group	1348	62	779	41	458	21

The ratio between the number of proton tracks in the 2nd group and the number in the lst group is 1:10. In order to find the ratio between the number of photons with an energy of 3.20 MeV and those with an energy of 2.62 MeV emitted in the same time interval by the MsTh source, it is necessary to take into account the dependence of the cross-section of ^{9}Be nuclei for the reaction $^9\mathrm{Be}(\gamma, n)\,^8\mathrm{Be}$ on the γ-ray energy as well as the dependence of the cross-section for elastic scattering of neutrons by protons upon the energy of the former. At the same time the small difference in absorption and scattering of photo-neutrons of the energies 0.83 MeV and 1.36 MeV by lead was not taken into consideration. The resulting intensities ratio of the lines at 3.20 MeV and at 2.62 MeV was evaluated as being about 1:12.

The histogram in Fig. 3 shows the results of photographing the recoil tracks. 1179 photographs were taken and these gave 530 tracks suitable for measuring. The lengths of the tracks were measured by projecting the negatives onto a screen and comparing the lengths of the proton tracks with that of a scale which was simultaneously photographed. Only tracks which were inclined to the direction of the neutron beam by not more than 45° and which were placed approximately in a hori-

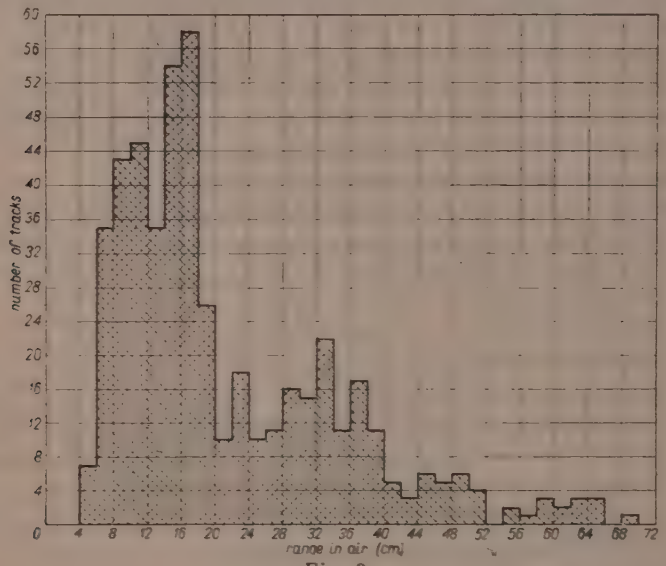

Fig. 3

zontal plane were measured. In Fig. 3 two groups of proton recoil tracks with lengths around 17 mm. and 37 mm. corresponding to the γ-ray energies 2.62 MeV and 3.20 MeV, may be distinguished. This histogram contains also α-ray tracks of radon, thoron and probably even actinon, all of them exceeding 37 mm. The purpose of producing Fig. 3 is solely to show the existence of the γ-radiation with an energy of about 3.20 MeV emitted by ThD.

The author wishes to express his gratitude to Professor H. Niewodniczański for suggesting this problem and for giving much helpful advice in the course of the work.

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CHEMISTRY

Titanic Acid Gel as an Adsorption Layer in Potentiometric Chromatography

by

B. KAMIEŃSKI and I. MAZUR

Communicated by B. KAMIEŃSKI on April 5, 1955

The smaller the volume of the electrolyte dividing the indicator electrode from the reference electrode, the greater is the influence of traces of external substances upon the electrode potential. The minimal mass of the electrolyte is obtained when the volume is reduced to a thin layer of silicic acid gel. The layer adsorbs traces of gases contaminating the atmosphere, and the electrode potential changes rapidly. The smaller the volume of the electrolyte, the greater is its resistance, and since layers of silicic acid are weak conductors of electricity, an electrometer of small capacity must be applied when checking the potential of the electric micro-element.

This sensitive electrode is composed of a thin metallic wire fused into a glass rod. The ground cross-section of the rod discloses the end of the wire which serves as an electrode. A thin silica gel layer is put on the cross-section of the wire and forms the electrolyte as described in previous papers by B. Kamieński [1]—[6]. The element which may be composed of a platinum, or antimony, wire and a reference electrode (e. g. a calomel electrode), provides an efficient apparatus for measuring the hydrogen ion concentration of the solution flowing out of the adsorption tube or for indicating slight traces of gases contaminating the atmosphere [2], [3], [7].

On the basis of the theory of adsorption [8], [9], it may be supposed that the gel of titanic acid, on account of its greater dielectric constant, will adsorb gases from the air more readily than the gel of silicic acid. The principles of electrostatics make it possible to calculate the change of energy accompanying the adsorption from air (or vacuum) of an electric charge e contained in a sphere of diameter a into a phase of a dielectric constant K_s (silicic acid gel) or into another phase of a dielectric constant K_t

(titanic acid gel). The change is

$$\frac{2e^2}{a} - \frac{2e^2}{aK_s} = \frac{2e^2}{a}\left(1 - \frac{1}{K_s}\right),$$

or

$$\frac{2e^2}{a} - \frac{2e^2}{aK_t} = \frac{2e^2}{a}\left(1 - \frac{1}{K_t}\right).$$

Subtracting the equations we get

$$\frac{2e^2}{aK_t} - \frac{2e^2}{aK_s} = \frac{2e^2}{a}\left(\frac{1}{K_t} - \frac{1}{K_s}\right),$$

which corresponds to the energy of transferring the electric charge (ion) from the silicic acid gel to the titanic acid gel. Taking into account the dielectric constant of titanic acid which is much greater than that of silicic acid (rock crystal has two constants, viz., 4.69 and 5.06, rutile 89 and 173), we may suppose that the gel of titanic acid will be a more efficient adsorbent. The antimony electrode (antimony wire fused into a glass rod) was covered with a very thin layer of dried titanic acid gel and the action of the electrode compared with another antimony electrode covered with a thin layer of silicic acid gel as described in former papers.

The titanic acid gel was prepared in the following way: 1 g. of titanic acid was mixed with 10 c. cm. of hydrochloric acid (sp. gr. 1.19). After 10 days the clear part of the solution was analysed. One c. cm. of the titanium chloride solution contained 9.43 mg. of titanic dioxide. A small quantity of the solution was necessary to carry out hydrolysis and to dialyze titanium chloride, and one fourth of a cubic centimetre was diluted with distilled water to the volume of 50 c. cm.; 16 c. cm. of the dilute solution were dialyzed for 15 days by changing the water twice daily. The meniscus of the solution was kept higher than that of the distilled water.

After 15 days no chloride ions were present and the volume increased to about 21 c. cm. Some of the titanic acid conglomerated in a sediment and milling of the colloidal solution was necessary before applying a drop to the end of the antimony electrode; the drop dried at room temperature, and a thin layer of titanic acid gel was formed at the bottom end of the glass rod antimony electrode. The reference electrode was an antimony electrode wetted with a solution of 0.1 N potassium chloride. This element was very sensitive to different gases. In this case the vapours of acetic acid were used. A square sheet of filter paper, 2 cm. × 2 cm., was saturated with acetic acid and placed at the bottom of a glass tube 7 cm. high and 4 cm. in diameter. The end of the electrode was placed at a distance of 6 cm. from the filter paper lap. The potential of the electrode changed rapidly and equilibrium was established after some time. When the potential was stabile, ammonium carbonate (1 gram) was put into

the glass tube in place of the saturated filter paper. The potential changed again, but in a reverse direction.

The purpose of the experiment was to ascertain whether a layer of titanic acid gel is a more efficient adsorbent than a layer of silicic acid gel. The mass of the layer was very small, as may be realized when we consider that only one drop of the solution was applied to the end of the glass rod electrode and dried to form the gel. One drop of silica gel contained $5.7.10^{-5}$ g. of silica whereas one drop of titanium gel contained $2.8.10^{-7}$ g. of titanium dioxide. Figs. 1 and 2 show

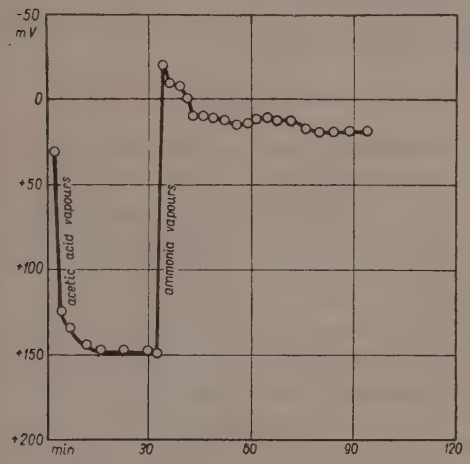

Fig. 1.

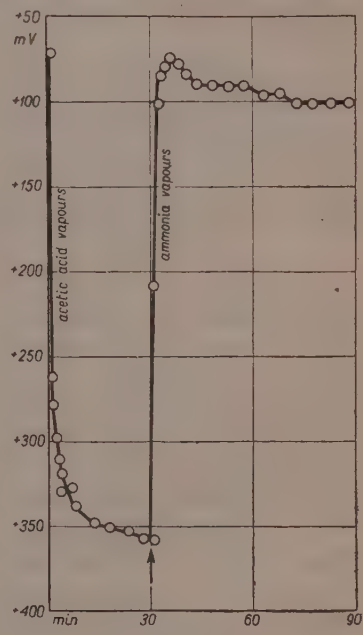

Fig. 2.

the changes in the potential millivolts on the ordinate, time on the abscissa) when silica gel (Fig. 1) and titanic acid (Fig. 2) were applied. The changes observed in the potential were greater when a layer of titanic acid gel was applied, as may be seen in Fig. 2; when the silica gel was applied to the electrode, the device became less sensitive, as may be seen in Fig. 1. The respective changes in the potential were 170 mV (Fig. 1), and 280 mV (Fig. 2). The above measurements confirm the supposition in regard to the influence of the dielectric constant. In the respective gels the mass of titanic dioxide was about 200 times smaller than the mass of silica but, in spite of this fact, the titanic gel was a more efficient adsorbent.

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CHEMISTRY

The Rôle of Vanadyl Sulphate in the Process of Catalytic Oxidation of Sulphur Dioxide

by

S. WEYCHERT and J. LEYKO

Presented by S. BRETSZNAJDER on April 13, 1955

Investigation of the synthesis and decomposition of vanadyl sulphate was carried out by the flow method in a temperature ranging from 360-630°, using a thermobalance. The reagents were chemically pure V_2O_5 and air containing $10-45\%$ SO_2.

In some experiments the gases were first passed through another furnace containing a vanadium catalyst, so that some of the SO_2 underwent conversion to SO_3 before the gases reached the reactor proper. In some of the last experiments V_2O_5 was reduced by pure SO_2.

The progress of the synthesis or decomposition of $VOSO_4$ was followed by reading the deflections of the thermobalance; the degree of conversion of SO_2 was determined by analysis of the gases leaving the reactor.

It was found that not all of the V_2O_5 present could be converted to $VOSO_4$, even if equilibrium conditions were exceeded. The degree of conversion of V_2O_5 continually rose with the temperature.

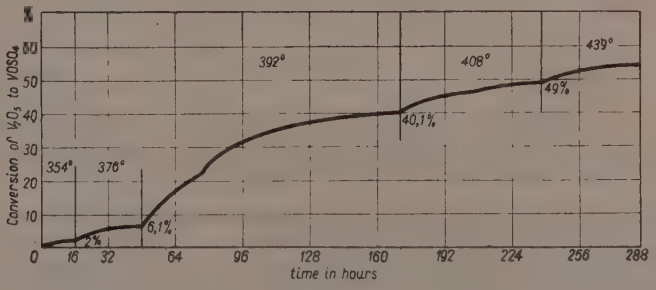

Fig. 1.

Only when the temperature exceeded a certain value did the decomposition of $VOSO_4$ begin. This critical point depended but slightly on

the velocity and composition of the gases and on the conditions under which synthesis of $VOSO_4$ had been previously carried out. Decomposition proceeds gradually with increasing temperature. At 10 – 20° above the initial temperature the decomposition is complete. The diagrams on Figs. 1 and 2 illustrate the course of one of the experiments of this type.

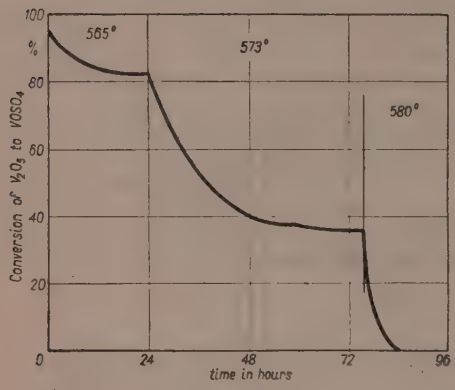

Fig. 2.

We observed that at lower temperatures reaction took place in the whole mass of V_2O_5, as shown by the uniformly distributed greenish tint. In spite of this, however, not all of the V_2O_5 present underwent conversion. On the contrary, when synthesis was carried out at higher temperatures, or when a partial decomposition of previously formed $VOSO_4$ was achieved, we obtained differently coloured pellets, their colour depending on their position in the bed (those placed nearer the gas-inlet were amber-coloured (V_2O_5), the others – green ($VOSO_4$).

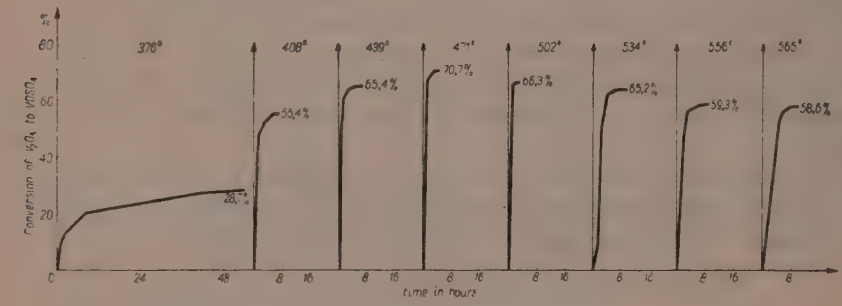

Fig. 3.

The course of some experiments on the synthesis of $VOSO_4$ from V_2O_5, SO_2 and O_2 at constant temperatures is illustrated by the diagrams in Fig. 3.

Curves are drawn from left to right according to the increasing temperatures of successive experiments.

It is evident from the diagram that the degree of conversion of V_2O_5 to $VOSO_4$ was at first higher in each successive experiment, but after passing a maximum became lower and lower as the temperature was raised further.

In each of these experiments the rate of $VOSO_4$ formation was at first very high, being in fact of the same order as the rate of oxidation of SO_2 to SO_3. However, when a certain degree of conversion was achieved, the reaction slowed down and very soon practically came to a standstill.

The initial rate was highest in experiments 2, 3, 4 and 6. In experiments performed at the lowest (1) and the highest (7 and 8) temperatures, it was visibly lower.

The numerical values of initial velocities are given in Table I.

TABLE I

The rate of $VOSO_4$ synthesis in experiments 1—8

No. of exp.	1	2	3	4	5	6	7	8
Temperature °C	376	408	439	471	502	534	556	565
Rate of synthesis ml. SO_2/hr. 1 g. V_2O_5	24.2	130	142.7	148.2	148.9	123.3	79.2	30.2

When gases were used, which had been already partly oxidised, it was possible to convert the whole mass of V_2O_5. However, this was achieved only when the equilibrium conditions for $VOSO_4$ stability were markedly exceeded.

We also found that the rate of synthesis of $VOSO_4$ was considerable, being not lower than that of the catalytic oxidation of sulphur dioxide. Under the conditions of our experiments the synthesis rate was nearly constant within a large range of temperatures. In Table II are given the numerical values of reaction rates corresponding to experiments conducted at different temperatures.

TABLE II

The rate of $VOSO_4$ synthesis with previous oxidation of SO_2

Degree of conversion of inlet-gases	20%	20%	20%	20%
Temperature °C.	376	439	534	565
Rate of synthesis ml. SO_2/hr./1 g. V_2O_5	156.6	148.1	151.6	75.5

In the experiments on reduction of V_2O_5 when pure SO_2 was used, it was observed that at lower temperatures $VOSO_4$ was at first formed. Later, in the course of the experiment it underwent decomposition to V_2O_4. At higher temperatures V_2O_4 was formed directly.

In both cases the reaction rate was low.

Conclusions

1. For the formation of vanadyl sulphate from V_2O_5 and gases containing SO_2 and SO_3 or SO_2 and O_2 the equilibrium conditions must be considerably exceeded, especially at lower temperatures.

Contrary to other authors, we have shown the equilibrium conditions to be quite inadequate for $VOSO_4$ synthesis. It is not certain if this is caused by the extremely high decomposition pressures of the vanadyl sulphate nuclei, which are formed on the surface of the dispersed and highly active phase of V_2O_5.

2. The rate of formation of $VOSO_4$ from V_2O_5 and from gases containing SO_2 and SO_3 is very high. In gases containing SO_2 and O_2 the rate is also high, being of the order of the rate of catalytic oxidation of SO_2. This is in direct contradiction to Boreskov's statement, that the rate of $VOSO_4$ formation is low.

3. It was confirmed that the reaction of V_2O_5 with SO_2, and O_2, giving $VOSO_4$ as a product, did not go on to completion, in spite of the fact that the conditions corresponded to equilibrium.

This may be explained by the fact that the conditions necessary for $VOSO_4$ synthesis can be realised only when a high degree of SO_2 to SO_3 conversion is achieved.

At low temperatures this is realised only inside the capillaries and at high temperatures in those layers of reacting material which are far from the gas-inlet.

The establishement of these facts should have a bearing on the methods and results of calculation of the ignition temperature of the contact process.

4. Further investigation is needed to explain the causes of the very high rate of $VOSO_4$ formation during the contact process. This rate probably exceeds the rate of the whole process of SO_2 oxidation (ending in desorption of SO_3).

5. The reduction of V_2O_5 to V_2O_4 proceeds very slowly even at high temperatures.

At low temperatures it is accompanied by the formation of $VOSO_4$. As this reaction is controlled by the former one, its rate is also very low.

The assumption that the formation of $VOSO_4$ which accompanies the reduction of V_2O_5, proceeds with no previous SO_3 desorption leads to the conclusion that the rate of synthesis of $VOSO_4$ as a whole should be controlled by the first stages of reduction.

It follows that the rate of the first stages of reduction is very low. This, in turn, excludes the possibility that reduction of V_2O_5 (not even in its first stages when SO_3 has not yet been desorbed and the proper crystalline phase of V_2O_5 formed) could be one of the stages of the usual contact process.

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CHEMISTRY

On the Distillation Anomalies Observed in Mixtures of Components Forming Ternary Positive-Negative Azeotropes and Zeotropes. I

by

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1. Observations of Ewell and Welch. In 1945 some interesting phenomena were described by Ewell and Welch [1] with regard to ternary systems in which two of the components formed with each other a binary negative azeotrope and with the third, a ternary zeotropic or azeotropic mixture. Owing to the shape of the tridimensional boiling temperature isobar ($p = 1$ atm.), a top-ridge curve divided the Gibbs triangle of concentrations into two parts. Only one system, that composed of chloroform, acetone and methanol, formed a positive-negative azeotrope. The latter still represents a unique example of a ternary saddle azeotrope which does not contain water. According to the classification suggested by one of us [2], it belongs to the first group, and its formation is due to the differences in the van der Waals forces acting between the molecules of the three components.

The anomaly taking place in the course of a fractional distillation of mixtures, forming systems characterised by the presence of a top-ridge condensation temperature isobar consists in the forming of several distillation levels, one of which, found in the middle or not far from the last level, is characterised, by a lower condensation temperature than that of the two neighbouring ones.

Several years later Lang [3] confirmed the existence of these anomalies. In both the above-mentioned papers, however, no observations were made with regard to the changes in composition of the fractions collected and of the bottom products remaining after each successive removal of the distillate. Another paper of Haase and Lang did not contain either sufficient data [3]. Because of that, no graphical presentation could be made of the gradual changes in the composition of the fractions collected and of mixtures found in the flask and in the column.

2. Behaviour of Ternary Positive-Negative Azeotropes Belonging to the Second Group. The purpose of the present investigation was to examine the behaviour of a new group of ternary positive-negative azeotropes formed by a weak organic (acetic) acid, *A*, a weak organic (pyridine) base, *P*, and a hydrocarbon, *H*. In our case this last was the normal octane. Starting with 1952, a relatively large number of ternary saddle azeotropes of this kind were examined in our laboratory and some of them were described [4], [5], [6], [7]. They did not contain water and the negative azeotrope [(−)A, P] was formed as a result of the chemical acting forces of an acid and a base.

For the first example there was chosen a saddle azeotrope in which the top-ridge condensation temperature isobar lay not far from the line *HC*, as shown in Fig. 1. In addition, the point $(\pm)Az$, corresponding to the composition of the ternary saddle azeotrope, is located closer to point *H* and somewhat higher than point M_P, which lies also on the top-ridge curve but corresponds to the maximum convexity of the top-ridge line. Its location is higher compared with other mixtures represented graphically by the same top-ridge curve HM_PC.

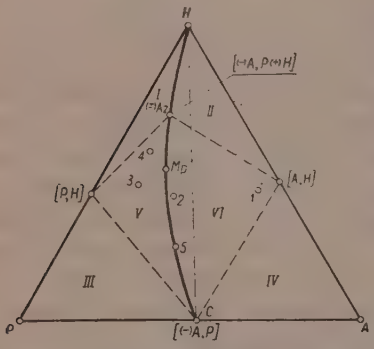

Fig. 1. Distillation fields of a ternary positive-negative azeotrope $[(-)A, P(+)H]$; all of them, except *V* and *VI*, form triangles. Fields *V* and *VI* are separated by the top-ridge curve $H(\pm)Az\ M_PC$. It is assumed that section $H(\pm)Az$ is practically a straight line.

There is another factor which exerts an influence on the distillation anomalies in the case of the saddle azeotropes examined in our laboratory. It consists in the large variety in shapes of the top-ridge condensation temperature isobars and in the locations of point $(\pm)Az$, corresponding to the composition of the ternary positive-negative azeotropes. In this investigation we chose the case graphically presented in Fig.1.

3. Distillation Curves and Material Balance of Distillate Fractions and Mixtures Remaining in the Flask and in the Column. In Fig. 1, points 1, 2, 3, 4 and 5 represent the respective compositions of the initial mixtures submitted to fractional distillation. All details concerning these five distillations, including the analytical methods used for determining the compositions of the numerous fractions collected, will be published elsewhere. In this paper are described the most interesting experiments observed in the course of fractional distillation of the mixtures represented by points 4 and 5.

The distillation curves are shown in Figs. 2*a* and 2*b*. The concentration triangles are presented in Figs. 3*a* and 3*b*; in each of them curves *I*

and *II* correspond to the compositions of the fractions collected (*I*) and of the remaining material (*II*) found in the flask and in the distilling column.

It should be noted that point 4 lies between points $(\pm)Az$ and M_P. Point 5 is located on the top-ridge curve below M_P.

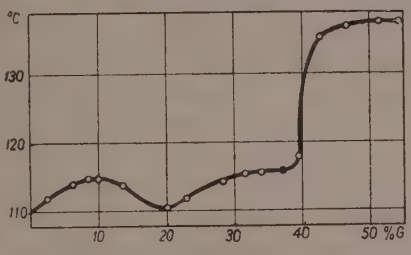

Fig. 2*a*

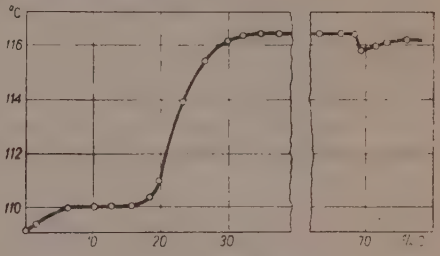

Fig. 2*b*

Figs. 2*a* and 2*b*. Distillation curves of mixtures 4 and 5. No horizontal level corresponding to the boiling temperature of the positive azeotrope $[P, H]$ is found on curve 2*b*. A pronounced temperature decrease corresponds to the moment when the total reflux has been established in the column.

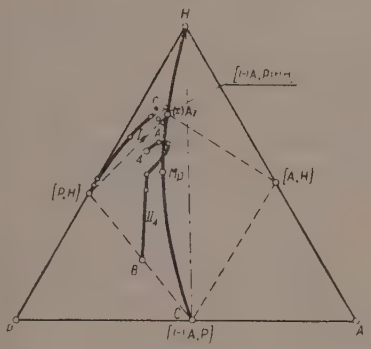

Fig. 3*a*

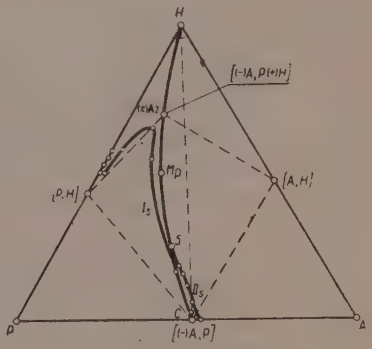

Fig. 3*b*

Figs. 3*a* and 3*b*. The compositions of the fractions collected when distilling mixtures 4 and 5, are shown on curves I_4 and I_5, and those of the bottom mixtures found in the flask and in the column (hold-up) are represented by curves II_4 and II_5.

At the beginning of the fractional distillation of mixture 4 the positive binary azeotrope $[P, H]$ was collected in the receiver. Section *A* on the "bottom product curve" II_4 is practically a straight line. In the next stage the fractions collected (section *C* on curve I_4) have the composition equal or nearly equal to that of the ternary saddle azeotrope $(\pm)Az$. The flat surface of this part of the tridimensional isobar, as well as the analytical errors in determining small amounts of the fractions collected must be considered as responsible for the dissipation of the points lying in the flat regions. At this stage of the distillation, the com-

position of the mixture found in the flask and in the hold-up of the column underwent changes, so that at the end of the distillation point B was reached. This point lies practically on the straight line connecting point $[P, H]$ with $[(-)A, P]$. At that moment there was not enough liquid to continue the distillation. One may assume, however, that a "straight line" distillation would have been followed, associated with a new drop in the condensation temperature. This would be associated with the distillation of azeotrope $[P, H]$ and afterwards with that of the negative one.

The fractional distillation of mixture 5 has been associated with the particularities shown in the graph of the two curves I_5 and II_5 (Fig. 3*b*). No horizontal levels were found in the course of the fractional distillation represented by the curve in Fig. 2*b*. Instead, a series of middle fractions appeared, which corresponded to mixture of $[P, H]$ and $(\pm)Az$ with an excess of the positive azeotrope in the first stage, and a decrease in the concentrations of $[P, H]$ in the second stage. Afterwards, the saddle azeotrope $(\pm)Az$ was collected and at the end of the distillation a mixture very similar to the composition of the negative azeotrope was found in the distillate.

The shape and the location of curve II_5 show the compositions shifting towards the left side of the main line HC. This corresponds to the higher concentrations of acetic acid compared with the last sections of curve I_5. It is probable that some thermal decomposition of the liquid in the flask was responsible for this shifting. It may be, however, that during that part of the distillation a new anomaly had appeared so that more pyridine passed into the receiver during the last stages of the distillation. Owing to this, both curves I_5 and II_5 have been shifted to the left side of the triangle.

It should be noted that a relatively large section of the distillation curve I_5 practically merged with the top-ridge condensation temperature isobar HM_PC.

We wish to express our thanks to W. Michałek for helping us with the distillation experiments and with the analytical work.

Summary

1. The recently investigated new group of ternary, positive-negative azeotropes, containing a weak organic acid, a weak organic base and a hydrocarbon, shows distillation anomalies similar to those found for a known saddle azeotrope composed of chloroform, acetone and methanol (*I*).

2. The saddle azeotropes $(\pm)Az$, containing acetic acid, A, pyridine, P, and normal octane, H, contains less pyridine than the mixture represented by point M_P, which lies on the top-ridge isobar and corresponds to the mixture characterised by its maximum convexity. The top-ridge curve is located closer to the main line HC as compared with azeotrope $[P, H]$.

3. The material balance of the fractions collected (curves I_4, I_5) and of the mixtures remaining in the flask and in the column (curves II_4, II_5) was presented graphically in Figs. 3*a* and 3*b* for the two most interesting cases (points 4 and 5 in Fig. 1)

4. In the case of the fractional distillation of the mixture represented by point 4, it was found that at the end of the distillation the bottom product in the flask corresponded to a mixture of the negative azeotrope [(—)A, P] and the positive one, [P, H]. The last section of curve II_4 is a straight line inclining under an acute angle to the main line HC.

5. The fractional distillation of mixture 5 (Fig. 1), represented by point 5 lying lower than M_P showed abnormal distillation phenomena in the first stage of the distillation. At the end, both the last fractions collected (I_5) and the mixtures found in the flask after each partial distillation (II_5) were shifted towards higher concentrations of acetic acid (A) as compared with the content of A in the negative azeotrope formed by A and P. It may be that this phenomenon was associated with some thermal decomposition of the residue in the flask.

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GEOLOGY

A New Quantitative Classification of Clastic Rocks

by
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Presented on February 24, 1955

Statistical studies have shown that the earth's sedimentary crust is composed primarily of such elements as clay — 82%, sands — 12%, and carbonates — 6% [1]. Particular quantitative combinations of these minerals are characteristic for certain definite kinds of clastic rock. In the course of geological surveys many such groups have been so far recognised, and their significance and importance as historical evidence are constantly increasing.

Today the classification of the clastic elements in the crust of the earth is a problem of ever growing importance, as was once the case with igneous rocks. It is a question of arranging in an orderly way the abundant material already identified, and although the significance of this problem has been recognized, no satisfactory arrangement of the material has yet been advanced, in spite of the numerous attempts undertaken in various research centres [2] [3] [4]. The reason for such a state of affairs is twofold: on the one hand, it lies in the nature of the material to be classified, while on the other, it is connected with the modern requirements of the natural sciences which stress, and rightly so, that any classification in this field should be based on quantitative analysis, since only such analysis can supply historic and genetic evidence.

It is also, of course, evident that the classification of clastic rocks should be based primarily on a quantitative analysis of the minerals of which the rocks are composed. In the case of igneous rocks this method has not only proved most successful, but is now the generally accepted one. It seems, therefore, most appropriate and natural that this course should be followed also when attempts are made to classify clastic rocks. Such attempts should take into account the fact that the most common kinds of rocks are composed of clay, sand (broadly speaking), and carbonates (calcite, dolomite).

Theoretically, all quantitative combinations of these three elements are possible in the composition of the sedimentary rocks here discussed.

Not all combinations, however, characterise some particular environment. The principal factors in the formation of groups are the morphological features of the sedimentary basin and the corresponding gravitation conditions. Thus, reciprocally, the quantitative combination of the elements in the clastic rocks should characterize the morphological features as a function of these features. From what has been said, therefore, it follows that an adequate classification system would include the characteristic features of the sedimentary enviroment.

In Poland, the first attempt at working out a classification system conforming to the required standards was made by K. Smulikowski [5], who divided clastic rocks into groups represented by corresponding and quantitatively defined sectors of a concentration triangle — clay, carbonates, and sand being placed at the vertices of this triangle. The triangle, devised by Smulikowski, greatly resembles Johannsen's model [6] worked out for the classification of igneous rocks.

In the present paper the author presents a new model of the concentration triangle, in which none of the three rock components (clay, sand, or carbonates) is privileged as the basis for classification. Such a procedure is consistent with the natural conditions. The three mineral elements from sedimentary rocks are placed at the vertices of the triangle. In "pure" clays, sands (sandstones and quartzites), and carbonates (limestones and dolomites) — the monomineralic rocks actually existing in the earth's crust — there are undoubtedly factors which make a definition of the characteristic features of their environment possible. For classification purposes the "purity" of the components is reduced to a conventionally accepted value equal to a 10% content of the other components (Fig. 1, sectors 1, 2, and 3). Consequently, the sides of the triangle correspond to the two-mineral rocks: sand-clay, clay-carbonates, and carbonate-sand (Fig. 1, sectors 4 to 12). According to the conventional agreement just made, the presence in these sectors of up to 10% of the third component is accepted in the course of identification. The most significant parts of the sides of the triangle seem to be those which lie half way between two vertices; there, rocks are grouped with equal or nearly equal quantities of two of the components (Fig.1, sectors 5, 8 and 11). For this reason they are consistently marked out in the projection in all three outer bands of the triangle.

Outlined in the central area of the triangle is sector No. 22, which comprises rocks with equal, or approximately equal, quantities of all three components. This is the sector of marly rocks.

The definition of marly rocks is, unfortunately, one of the least precise in geology and petrography, hence, by the introduction of a strict quantitative definition for the mineral ratios of the marls some precision will at once be achieved. Moreover, the sedimentary enviroment,

where such rocks are formed, becomes defined more accurately than was possible previously.

Adjacent to the area representing marls are three other triangles (Fig. 1, sectors 19, 20, and 21) with their vertices pointing to the vertices of the outer triangle. In these are grouped marly rocks in which clayey, sandy, or limy components are predominant. This is stressed by the names applied to these sectors, viz., sandy marls (19), clayey marls (20), and limy marls (21).

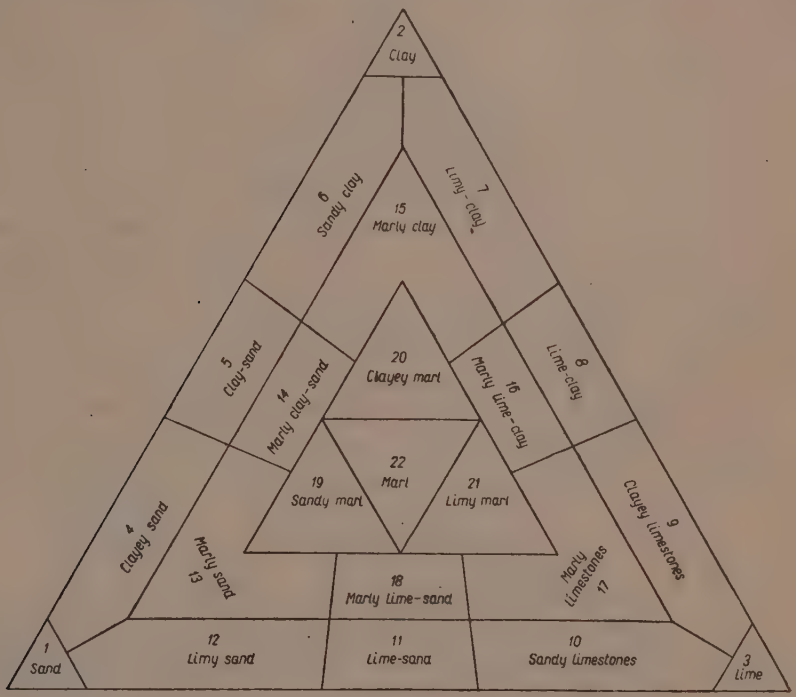

Fig. 1

Finally, between the outer bands already referred to and the inner triangles representing marly rocks three intermediate bands are marked out; these differ from the outer bands by a conventionally accepted $10 - 20\%$ content of the third component (Fig. 1, sectors 13 to 18). The whole of the triangle is thus divided into three concentration triangles, one inside the other. The consistent principle of classification is, therefore, differentiation of the three quantitative variants: the monomineral, two-mineral, and three-mineral materials. Obviously, the decimal system agreed upon here is only conventional, and quite certainly the rocks found in nature have no need to conform to it. Hence, the classification here described is devised for practical purposes and is intended to faci-

litate the distinction of groups in the clastic rocks and, what is even more important, to help those whose job it is to study such rocks.

By applying the method here described, all clastic rocks are divided into four classes, while every class is subdivided into orders, the total number of which is 22.

The general pattern of the classification is illustrated in the appended table, and the two concentration triangles in Figs. 1 and 2.

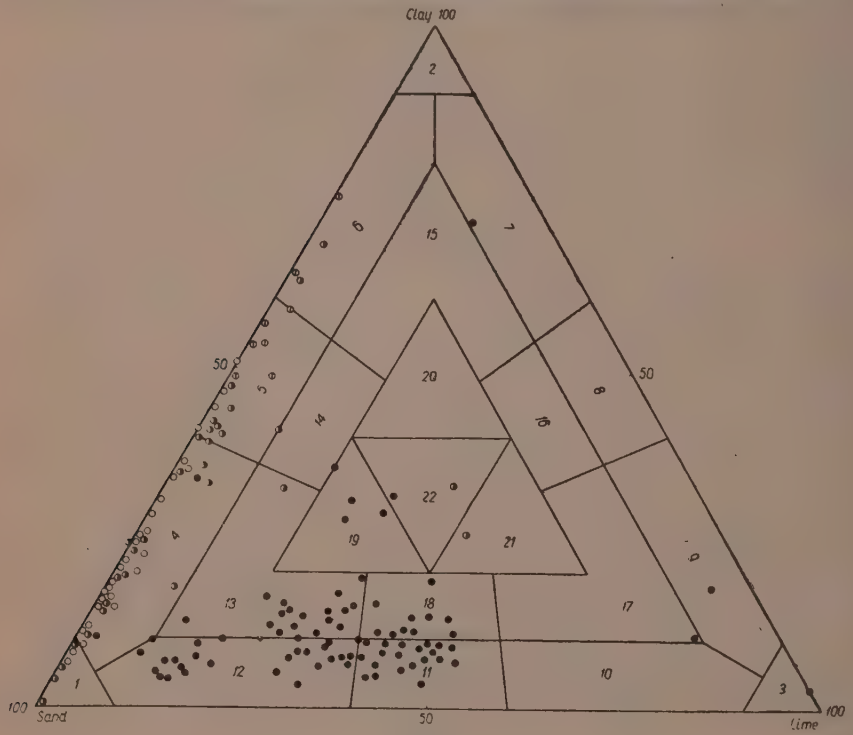

Fig. 2

The classification is based — as has already been stated — on a quantitative mineral analysis of the clastic formations.

In research work on clastic rocks no accurate methods have so far been devised for carrying out the necessary analyses.

The microscopic (planimetric) techniques, very useful for research on the composition of igneous rocks, cannot always be applied in the case of clastic rocks, because of the often undefined nature of the component elements. However, the analytic technique developed recently by the author and defined as "thermal analysis" [7] is very suitable for this purpose. This technique enables us to determine with a considerable degree of accuracy the mineral composition of the rocks under

CLASSIFICATORY TABLE

(S — Sand, Cl — Clay, Cb — Carbonates)

Class I. Monomineral Rocks

(Chief component 100—90%, others 0—10%)

Order 1.	Sands (sandstones, quartzites etc)	S_{100-90}
Order 2.	Clay (clay, shale etc)	Cl_{100-90}
Order 3.	Carbonates (limestones, dolomites)	Cb_{100-90}

Class II. Two-mineral Rocks

(Two chief components 100—90%, the third 0—10%

Order 4.	Clayey sand (sandstones etc)	S_{90-60} Cl_{5-40}
Order 5.	Clay-sand (loam)	S_{60-40} Cl_{40-60}
Order 6.	Sandy clay	S_{40-5} Cl_{60-90}
Order 7.	Limy-clay	Cl_{90-60} Cb_{5-40}
Order 8.	Lime-clay (dolomite, clay)	Cl_{60-40} Cb_{40-60}
Order 9.	Clayey limestones (dolomites)	Cl_{40-5} Cb_{60-90}
Order 10.	Sandy limestones (dolomites)	S_{40-5} Cb_{5-90}
Order 11.	Lime-sand	S_{60-10} Cb_{40-60}
Order 12.	Limy sand (sandstones etc)	S_{60-90} Cb_{40-5}

Class III. Three-mineral Rocks

(Two components 80—20%, the third 10—20%)

Order 13.	Marly sand (sandstones etc)	S_{80-50} Cl_{10-35} Cb_{10-35}
Order 14.	Marly clay-sand	S_{55-35} Cl_{55-30} Cb_{10-20}
Order 15.	Marly clay	S_{10-35} Cl_{50-80} Cb_{10-30}
Order 16.	Marly lime-clay	S_{10-20} Cl_{30-55} Cb_{35-55}
Order 17.	Marly limestones	S_{10-35} Cl_{10-35} Cb_{50-80}
Order 18.	Marly lime-sand	S_{35-55} Cl_{10-30} Cb_{30-55}

Class IV. Marly Rocks

(Three components 20—60%)

Order 19.	Sandy marl	S_{40-60} Cl_{20-40} Cb_{20-40}
Order 20.	Clayey marl	S_{20-40} Cl_{40-60} Cb_{20-40}
Order 21.	Limy marl	S_{20-40} Cl_{20-40} Cb_{40-60}
Order 22.	Marl	S_{20-40} Cl_{20-40} Cb_{20-40}

investigation, and is based on a quantitative estimation of the dehydration, oxidation, and dissociation processes which take place at various temperatures.

This method has been applied in hundreds of cases and has always proved most useful [8], [9], especially for purposes of comparison.

The value of the techniques introduced in this paper for obtaining a rational classification of clastic rocks is demonstrated here by plotting analytical results in the concentration triangle (Fig. 2). The material used for this purpose was obtained in the course of thermal analyses carried out on two series of clastic samples. In one of the series the samples came from known oligocene sandstones forming the geological profile of the Krosno district in the Carpathians. In the other series the same analytic techniques were applied to various kinds of sandy soils collected in the Cracow area; this series also included fractions of less than two microns (defined in soil science as "colloids") which have been analysed and reported separately.

The tabulated numerical results obtained analytically for the experimental material are not given, because the three-component parameters for every analysis can easily be deduced from the projection.

About 500 analyses were carried out on samples obtained from strata in the Krosno district, but not all the results were plotted in the triangle, as many of them coincided completely or very nearly so (black spots).

The soils from around Cracow (diluvial and alluvial) are indicated by the half-shaded circles. The empty circles correspond to the analytical results of alluvial soils from the Vistula valley in Goczalkowice, while the "soil colloids" are represented by the bisected circles.

Conclusions

When the distribution in the concentration triangle of the points representing the various kinds of detrital materials is considered carefully, the following conclusions may be drawn:

1. The analyses of the sandstones from the Krosno district give results grouped very closely together, chiefly in sectors 11, 12, 13 and 18. This indicates that, in principle, those materials correspond lithologically to the four systematically distinct orders known as lime-sands, limy sands, marly sands, and marly lime-sands. Some of the samples from the Carpathians correspond to sector 19, defined as sandy marls. One analytical result corresponds to order 22 (marl), two others to order 4 (clayey sand), one to order 7 (limy-clay), one to order 9 (clayey limestones), and, finally, one result corresponds even to order 3 ("pure carbonates").

In this manner the sedimentary environment in which these Flysh formations originated is defined by the quantitative mineral composition of the rock elements as shown by the concentration triangle. One out-

come of this investigation which seems especially significant is the singling out of elements from the whole series, which, by reason of their parameters, diverge markedly from the most common samples grouped in sectors 11, 12, 13, and 18. Such divergent elements may be considered as "indices" and may in future, have a practical application in the important problem of correlating the stratigraphic levels in the Carpathian oil-bearing series.

2. The points representing most of the analysed soils, alluvial soils, and "colloids" are closely grouped along the two-mineral sandclay sector. The drop in their fertility, as shown by the low level of calcium carbonate, is clearly visible. There is, however, one exception, viz., the diluvial soils from the Dolina Strążyska in the Tatra Mountains. The points representing these soils are grouped in the sectors corresponding to a high level of calcium carbonate (13, 14, 21, and 22). There is no doubt that because of this the soils are fertile (rendzines).

The analytical data presented here, obtained by thermal analysis based on the quantitative evaluation of the principal mineral components of the clastic rocks, fully confirm the value for research work on sedimentary rocks and their derivatives of the classification proposed in this paper.

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